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ON MATHEMATICAL INDUCTION*

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Introduction. According to modern standards of logical rigor, each branch of pure mathematics must be founded in one of two ways: either its basic concepts must be *defined* in terms of the concepts of some prior branch of mathematics, in which case its theorems are deduced from those of the prior branch of mathematics with the aid of these definitions, or else its basic concepts are taken as *undefined* and its theorems are deduced from a set of axioms involving these undefined terms.

The natural numbers, 0, 1, 2, 3, $\cdots$ are among those mathematical entities about which we learn at the earliest age, and our knowledge of these numbers and their properties is largely of an intuitive character. Nevertheless, if we wish to establish a precise mathematical theory of these numbers, we cannot rely on unformulated intuition as the basis of the theory but must found the theory in one of the two ways mentioned above. Actually, both ways are possible. Starting with pure logic and the most elementary portions of the theory of sets as prior mathematical sciences, the German mathematician Frege showed how the basic notions of the theory of numbers can be defined in such a way as to permit a full development of this theory. On the other hand the Italian mathematician Peano, taking *natural number*, *zero*, and *successor* as primitive undefined concepts, gave a system of axioms involving these terms which were equally adequate to allow a full development of the theory of natural numbers. In the present paper we shall examine the concept of *definition by mathematical induction* within the framework of Peano's ideas. In this development we shall presuppose only logic and the most elementary portions of the theory of sets; however, we shall find that our subject is greatly illuminated by the introduction of some of the terminology of modern abstract algebra, even though we do not presuppose any of this algebraic material as a basis of our proofs or definitions.

1. Models and the axioms of Peano. It will be convenient here to use the word *model* to refer to a system consisting of a set N, an element 0 of N, and a unary operation† S on N. A model $\langle N, 0, S\rangle$ will be called a *Peano model* if it satisfies the following three conditions (or axioms).

P1. *For all* $x\in N$, $Sx\neq 0$.

P2. *For all* $x, y\in N$, *if* $x\neq y$ *then* $Sx\neq Sy$.

P3. *If* G *is any subset of* N *such that* (a) $0\in G$, *and* (b) *whenever* $x\in G$ *then also* $Sx\in G$, *then* $G=N$.‡

* This article was translated into Russian and published in Matematicheskoe Prosveshchenie, No. 6, 1959. This English version appears with the consent of the editors of the Soviet publication.

† A unary operation on N is a function having N as its domain and having a range which is a subset of N. For any $x\in N$, we let Sx be the element of N which is the value obtained by operating on x with S.

‡ In the terminology of set-theory, P1 and P2 respectively express the conditions that 0 is not in the range of S, and that S is one-one. A subset G of N which satisfies condition (b) of P3 is said

If we consider the case when N^* is the set of natural numbers as we know them intuitively, 0^* is the number zero, and S^* is the successor operation (*i.e.*, the operation such that for any natural number x, S^*x is the next following number), then $\langle N^*, 0^*, S^* \rangle$ is an example of a Peano model. Of course it was this example which suggested consideration of the Peano axioms in the first place. Note, however, that there are many other Peano models, for example the system $\langle N', 0', S' \rangle$ where N' is the set of all positive even integers, $0'$ is the number two, and S' is the operation of adding two.

Condition P3 is called the *axiom of mathematical induction*. It will be useful to introduce the term *induction model* to refer to any model which satisfies this axiom. Thus we see at once that every Peano model is an induction model. But the converse is far from true.

For example, let N'' be a set containing a single element, let $0''$ be this element, and let S'' be the only possible unary operation on N'', *i.e.*, the operation such that $S''0''=0''$. Then clearly $\langle N'', 0'', S'' \rangle$ is an induction model, but it is not a Peano model since it does not satisfy Axiom P1. For another example let a_0 and a_1 be two distinct objects, and let N''' be the pair $\{a_0, a_1\}$ (*i.e.*, the set having a_0 and a_1 as its only elements). Let S''' be the unary operation on N''' with constant value a_1 (*i.e.*, $S'''x=a_1$ for all $x \in N'''$). Then $\langle N''', a_0, S''' \rangle$ is an induction model, but it is not a Peano model since it does not satisfy Axiom P2.

The reader may notice that the model $\langle N'', 0'', S'' \rangle$ satisfies P2, while $\langle N''', a_0, S''' \rangle$ satisfies P1. It then becomes natural to ask whether there are induction models which satisfy *neither* P1 nor P2. As it happens, there are none: any model which satisfies P3 must also satisfy either P1 or P2. A direct proof of this fact, using only the laws of logic and the elements of set theory, is rather troublesome to find—it is a task we leave for the enterprising reader. Presently we shall see that after the theory of definition by mathematical induction is established, the result can be obtained quite simply.

Now consider the following two statements.

P4. *If y is any element of N such that $y \neq Sx$ for all $x \in N$, then $y=0$.*
P5. *For all $x \in N$, $x \neq Sx$.*

We recognize that each of these is true for the Peano model $\langle N^*, 0^*, S^* \rangle$ of natural numbers as we know them intuitively, and in fact we can easily show that each one is true of *all* Peano models by deriving it from Axioms P1–P3. But there is an important difference between the two: the proof of P4 requires

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to be *closed* under S. If we regard $\langle N, S \rangle$ as an algebraic system, a subset G of N which is closed under S would be called a *subalgebra* of the system. Thus P3 expresses the condition that the only subalgebra of $\langle N, S \rangle$ which contains the element 0 is N itself. In algebraic terminology this condition is expressed by saying that the element 0 *generates* the algebra $\langle N, S \rangle$. It is also possible to regard the system $\langle N, 0, S \rangle$ itself as an algebraic system. In that case, in order to qualify as a subalgebra a subset G of N must contain 0 as well as be closed under S. Thus P3 expresses the condition that the only subalgebra of $\langle N, 0, S \rangle$ is N itself. Hence the system $\langle N, 0, S \rangle$ is generated by any one of its elements.

only Axiom P3, so that P4 holds for all induction models, but the proof of P5 requires Axiom P1 as well as P3, and our example $\langle N'', 0'', S'' \rangle$ shows that P5 does not hold for all induction models.

2. Operations defined by mathematical induction. The statements P4 and P5 are examples of true theorems about natural numbers, but we generally regard their mathematical content as quite trivial. In order to develop a richer theory it is essential to introduce additional concepts beyond the primitive notions of number, zero, and successor; in particular, we must define such central concepts as addition, multiplication, exponentiation, prime number, *etc.* Let us consider addition first, and inquire how it can be defined.

Addition is a binary operation on natural numbers; *i.e.*, it is a function $+$ which may act on any ordered pair $\langle x, y \rangle$ of natural numbers, the result of the action being again a natural number, $x+y$. Peano's idea was to define $+$ by means of the pair of equations

$$x + 0 = x, \tag{1.1}$$

$$x + Sy = S(x + y). \tag{1.2}$$

Of course on the basis of our intuitive knowledge of the operation of addition we recognize that these equations are true for all natural numbers x and y. But in what sense do the equations constitute a *definition* of addition? In particular, does the definition hold only for natural numbers, or for arbitrary Peano models as well? In order to get a clear answer to these questions we first consider a related but more general problem.

The introduction of an operation by means of the pair of equations 1.1 and 1.2 is an example of what is called *definition by mathematical induction.* To describe this concept in general terms we must consider a Peano model $\langle N, 0, S \rangle$ and in addition a second model $\langle N_1, 0_1, S_1 \rangle$ which, however, is not required to be a Peano model (or even an induction model). Being given these two models we say that the pair of equations

$$h0 = 0_1, \tag{2.1}$$

$$h(Sy) = S_1(hy), \tag{2.2}$$

defines (by mathematical induction) a function h: a function which maps N into N_1 and satisfies 2.1 and 2.2 for all $y \in N$. Again we may raise the question: In what sense do these equations define a function? The answer is provided by the following theorem.

THEOREM I. *No matter what Peano model $\mathfrak{N} = \langle N, 0, S \rangle$ we have, and no matter what model $\mathfrak{N}_1 = \langle N_1, 0_1, S_1 \rangle$ we start with, there exists a unique homomorphism of $\mathfrak{N}$ into $\mathfrak{N}_1$; that is, there exists one and only one function h mapping N into N_1 which satisfies* 2.1 *and* 2.2 *for all $y \in N$.*†

† As remarked in footnote ‡, pp. 323–324, a model $\langle N, O, S \rangle$ is an induction model if and only if 0 is a generator of the algebraic system $\langle N, S \rangle$. In algebraic terminology the content of Theorem I

Before attempting to prove this theorem let us see how it applies to the case of 1.1 and 1.2.

Let $\mathfrak{N}$ be an arbitrary Peano model $\langle N, 0, S\rangle$, and for each $x \in N$ let $\mathfrak{N}_x$ be the model $\langle N, x, S\rangle$. Applying Theorem I to the models $\mathfrak{N}$ and $\mathfrak{N}_x$ we see that for each $x \in N$ there will be a unique function h_x mapping N into itself such that the equations

$$3.1 \qquad h_x 0 = x,$$

$$3.2 \qquad h_x(Sy) = S(h_x y),$$

hold for all $y \in N$. From the existence and uniqueness of these functions h_x we shall show that there exists a unique binary operation of addition on $\langle N, 0, S\rangle$; for later purposes the reader should note that the argument by which we infer the existence and uniqueness of addition from the existence and uniqueness of the functions h_x is purely set-theoretical in character, and does not depend in any way on Axioms P1–P3. Let f be the binary operation on N whose value for any $x, y \in N$ is determined by the equation

$$4 \qquad fxy = (h_x y).$$

Using 4, 3.1, and 3.2 we see that f satisfies the equations

$$5.1 \qquad fx0 = x,$$

$$5.2 \qquad fx(Sy) = S(fxy),$$

for all $x, y \in N$. Furthermore, f is the *only* binary operation on N with this property. For suppose g is any binary operation on N satisfying

$$6.1 \qquad gx0 = x,$$

$$6.2 \qquad gx(Sy) = S(gxy),$$

for all $x, y \in N$. Then for each $x \in N$ let g_x be the unary operation on N such that

$$7 \qquad g_x y = gxy$$

for all $y \in N$. From 7, 6.1, and 6.2 we infer that for any $x \in N$ the equations

$$8.1 \qquad g_x 0 = x,$$

$$8.2 \qquad g_x(Sy) = S(g_x y),$$

hold for all $y \in N$. Comparing 3.1, 3.2 and 8.1, 8.2 we see that for each $x \in N$ we have $g_x = h_x$, since Theorem I assures us that the function h_x defined by 3.1 and 3.2 is unique. But if $g_x = h_x$ for each $x \in N$, 4 and 7 imply that $fxy = gxy$ for all $x, y \in N$. Since f and g are binary operations on N, we see that $f = g$ by the principle of extensionality.

We have thus inferred from Theorem I that if $\langle N, 0, S\rangle$ is any Peano model,

would be expressed by saying that in the class of all algebraic systems $\langle N, S\rangle$ the systems $\langle N, S\rangle$ derived from Peano models $\langle N, 0, S\rangle$ are *free*, and in fact are *freely generated by* 0.

there is a unique binary operation f on N satisfying 5.1 and 5.2 for all $x, y \in N$. This binary operation we call addition; if we denote it by the symbol "$+$", we see that 1.1 and 1.2 are simply another way of writing 5.1 and 5.2. In this way the definition of addition by 1.1 and 1.2 is explained by Theorem I.

We now take up the problem of proving this theorem.

Suppose, then, that $\langle N, 0, S\rangle$ is any Peano model, and that $\langle N_1, 0_1, S_1\rangle$ is an arbitrary model. We wish to show the existence of a unique function h mapping N into N_1 which satisfies 2.1 and 2.2 for all $y \in N$. Putting aside for a moment the question of uniqueness, an argument to show the existence of such a function h is sometimes given along the following lines.

Clearly (the argument goes), h is defined for 0, since $h0 = 0_1$ by 2.1. Furthermore, if h is defined for an element y of N then h is also defined for Sy since $h(Sy) = S_1(hy)$ by 2.2. Thus if we let G be the set of all those $y \in N$ for which h is defined, we see that (a) $0 \in G$, and (b) whenever $y \in G$ then also $Sy \in G$. Applying Axiom P3 we conclude that $G = N$. Thus h is defined for all $y \in N$.

At first sight this argument may seem convincing, but a moment's reflection will suffice to raise doubts. For in this argument we refer to a certain function h. But what is h? Apparently it is a function which satisfies 2.1 and 2.2. Recall, however, that the argument is designed to establish the existence of such a function; clearly, then, it is incorrect to assume in the course of the argument that we have such a function.

This objection may be thought at first to be simply a verbal matter which can be avoided by some minor rewording of the argument. Actually, however, there is something fundamentally wrong with the argument—for the only property of the model $\langle N, 0, S\rangle$ which it employs is Axiom P3! If this argument were essentially correct, it would follow that for any *induction* model $\langle N, 0, S\rangle$, and an arbitrary model $\langle N_1, 0_1, S_1\rangle$, there exists a function h, mapping N into N_1, which satisfies 2.1 and 2.2 for all $y \in N$. But this statement is simply false, as the following example shows.

Let $\mathfrak{N}'' = \langle N'', a_0, S''\rangle$ be the induction model considered in Section 1, in which N'' is the pair of two distinct objects a_0 and a_1, and $S''x = a_1$ for all $x \in N''$. Let T be the unary operation on N'' such that $Ta_0 = a_1$ and $Ta_1 = a_0$, and let $\mathfrak{N}_1$ be the model $\langle N'', a_0, T\rangle$. Now if Theorem I could be applied to the induction model $\mathfrak{N}''$ and the model $\mathfrak{N}_1$, there would exist a mapping h of N'' into itself such that

$$h a_0 = a_0, \tag{9.1}$$

$$h(S''y) = T(hy), \tag{9.2}$$

for all $y \in N''$. From 9.2 we compute that $h(a_1) = h(S''a_0) = T(ha_0)$, and so by 9.1, $h(a_1) = Ta_0 = a_1$. On the other hand, $h(a_1) = h(S''a_1) = T(ha_1)$ by another application of 9.2, and since we have already computed $ha_1 = a_1$ this shows that $h(a_1) = Ta_1 = a_0$. From $h(a_1) = a_0$ and $h(a_1) = a_1$ we get $a_0 = a_1$, contrary to our hypothesis that a_0 and a_1 are distinct. This contradiction shows that Theorem I does

not apply to the induction model $\mathfrak{N}''$, and shows *a fortiori* that any proof of this theorem must employ either Axiom P1 or P2, as well as P3; actually, as we shall see later, all three axioms must be employed. We may phrase this observation by saying that the *axiom* of mathematical induction does not itself justify *definitions* by mathematical induction.†

Proof of Theorem I. Let $\mathfrak{N}=\langle N, 0, S\rangle$ be any Peano model and let $\mathfrak{N}_1=\langle N_1, 0_1, S_1\rangle$ be an arbitrary model. A subset H of N is called a *segment* if $0\in H$ and if, whenever $Sx\in H$ then also $x\in H$. By a *partial function* let us mean a function j whose domain is some segment H, which has values in N_1, and which is such that

$$10.1 \qquad j0 = 0_1,$$

$$10.2 \qquad j(Sx) = S_1(jx)$$

for all x such that $Sx\in H$.

LEMMA 1. *Every element of N is in the domain of some partial function.*

Proof. Let G be the set of those elements of N which are in the domain of some partial function. It is clear that the set $\{0\}$ whose only element is 0 is a segment, since there is no x such that $Sx\in\{0\}$ according to Axiom P1. Furthermore, the function j whose domain is $\{0\}$ and whose value $j0$ is 0_1 is a partial function for the same reason. Hence $0\in G$.

Now suppose that y is any element of G, and let j be a partial function having y in its domain. Let H be the domain of j. If $Sy\in H$ then Sy too is in G; so let us consider the case where $Sy\notin H$. In this case let H' be the subset of N obtained from H by adding Sy as an additional element, and let j' be the function whose domain is H' and whose values are given by the following rule: if $x\in H$ then $j'x=jx$, and if $x=Sy$ then $j'x=S_1(jy)$. We shall show below that j' is a partial function, and hence that $Sy\in G$ in the case that $Sy\notin H$ (as well as in the contrary case considered above). Since G contains 0 and is closed under S, it will follow by Axiom P3 that $G=N$. By definition of G the lemma will therefore be proved.

Thus to complete the proof of the lemma it remains only to show that j' is a partial function. To this end, consider first the domain H' of j'; we have $H'=H\cup\{SY\}$. Since H is the domain of a partial function it is a segment, and hence $0\in H$, so that $0\in H'$. Further, if x is any element of N such that $Sx\in H'$, then also $x\in H'$ as we see by cases: if $Sx\in H$ then $x\in H$ because H is a segment, and if $Sx=Sy$ then $x=y$ by Axiom P2 so that $x\in H$ again (since we know $y\in H$). These considerations show that H' is a segment. Now $j'0=j0=0_1$, and $j'(Sx)=S_1(j'x)$ whenever $Sx\in H'$; the last equation is again established by cases, for if $Sx\in H$ then also $x\in H$ so that $j'(Sx)=j(Sx)$ and $j'x=jx$, while if $Sx=Sy$ then $x=y$ and so $j'(Sy)=S_1(j'y)$. Since j' has a domain which is a seg-

† This fact was clearly brought out by Dedekind in his famous book: Was sind und was sollen die Zahlen? (See "Bemerkung," paragraph 130, section 9.)

ment, and since it satisfies 10.1 and 10.2, it is a partial function. This completes the proof of Lemma 1.

LEMMA 2. *If j_1 and j_2 are partial functions and x is in the domain of each, then $j_1x=j_2x$.*

Proof. Let G be the set of those elements x of N such that $j_1x=j_2x$ whenever j_1 and j_2 are partial functions each of which has x in its domain. Clearly $0\in G$, since $j0=0_1$ for any partial function j. Now suppose x is an arbitrary element of G, and let j_1 and j_2 be any partial functions each of which contains Sx in its domain. Then $j_1(Sx)=S_1(j_1x)$ and $j_2(Sx)=S_1(j_2x)$; but since $x\in G$, $j_1x=j_2x$ and hence $j_1(Sx)=j_2(Sx)$. Therefore $Sx\in G$. Since $0\in G$ and G is closed under S, it follows by Axiom P3 that $G=N$. Referring to the definition of G we see that Lemma 2 is proved.

By combining Lemmas 1 and 2 we see that for any $x\in N$ there is one and only one $z\in N$ such that $z=jx$ for some partial function j. We let h be the function, with domain N, such that for any $x\in N$ the value hx is this unique $z\in N$. We claim that this function h satisfies 2.1 and 2.2 for all $y\in N$, and that it is the only function with domain N which has this property.

Clearly $h0=0_1$ since $j0=0_1$ for any partial function j. Furthermore, for any $y\in N$ there is a partial function j such that Sy is in the domain of j (by Lemma 1), whence we see that $h(Sy)=j(Sy)=S_1(jy)=S_1(hy)$. so that h satisfies 2.1 and 2.2 as claimed. Now if h_1 is any other function with domain N which satisfies 2.1 and 2.2 for all $y\in N$, then $h_1=h$. The reason is that N is clearly a segment, so that h and h_1 are both partial functions, whence by Lemma 2 we have $hx=h_1x$ for all $x\in N$, and therefore $h=h_1$ by the principle of extensionality. This completes the proof of Theorem I.

This proof is rather more involved than the simple argument sketched at first, but it has the advantage of being correct. The reader will notice that all of the Axioms P1–P3 were employed in the proof (in connection with Lemma 1).

The construction of h by means of partial functions, as described in this proof, is not the only means we know of obtaining this function. There is another method of constructing h which leads to a different proof of Theorem I. We will outline this other process of construction briefly, leaving the reader to supply details of the proof.

Being given a Peano model $\langle N, 0, S\rangle$ and an arbitrary model $\langle N_1, 0_1, S_1\rangle$, we consider subsets A of the product set $N\times N_1$, *i.e.*, sets A all of whose elements are ordered pairs $\langle x, y\rangle$ where $x\in N$ and $y\in N_1$. We call such a set A *regular* if $\langle 0, 0_1\rangle\in A$ and if, whenever $\langle x, y\rangle\in A$ then also $\langle Sx, S_1y\rangle\in A$. Clearly there are regular sets, for example $N\times N_1$ itself. It is easy to see that the intersection A^* of all regular sets A is itself regular. Now using Axioms P1–P3, which hold for $\langle N, 0, S\rangle$, we can show that for every $x\in N$ there is one and only one $y\in N_1$ such that $\langle x, y\rangle\in A^*$: this is the part of the proof where the reader will have to supply some detail. Having shown this, we define h to be the function with domain N such that for any $x\in N$, hx is the unique $y\in N_1$ for which $\langle x, y\rangle\in A^*$.

From the fact that A^* is regular it then easily follows that h is a homomorphism of N into N_1, as required by Theorem I. Finally, to show the uniqueness of h, we consider any homomorphism h_1 of N into N_1. Letting B be the subset of $N \times N_1$ such that $\langle x, y\rangle \in B$ if and only if $y = h_1 x$, we easily see that B is regular and hence $A^* \subseteq B$. From the fact that for each $x \in N$ there is only one $y \in N_1$ such that $\langle x, y\rangle \in B$, we then infer that $A^* = B$; and from this we easily conclude that $h_1 = h$. This completes our outline of the second proof of Theorem I.

3. Addition and multiplication in arbitrary induction models. As we have previously noted, we can infer from Theorem I the existence, in every Peano model, of a unique operation of addition (*i.e.*, a binary operation f which satisfies 5.1 and 5.2 for all $x, y \in N$). Actually, something more is true.

THEOREM II. *In every induction model there is a unique operation of addition.*

We cannot hope to prove this theorem by means of Theorem I, for the latter is not true of all induction models, as we have seen in Section 2. Instead, we proceed by means of the following lemma.

LEMMA. *If $\langle N, 0, S\rangle$ is any induction model, then for every $x \in N$ there is a unique unary operation h_x on N such that 3.1 and 3.2 hold for all $y \in N$.*

Proof. We first observe that for any $x \in N$ there can be at most one operation h_x satisfying 3.1 and 3.2. For suppose h_x and h_x' are both operations which satisfy these equations, and let G be the subset of N such that $y \in G$ if and only if $h_x y = h_x' y$. Clearly $0 \in G$, since $h_x 0 = x = h_x' 0$. Also G is closed under S, for if $y \in G$ (so that $h_x y = h_x' y$), we see that $h_x(Sy) = S(h_x y) = S(h_x' y) = h_x'(Sy)$ whence $Sy \in G$. Since Axiom P3 holds for $\langle N, 0, S\rangle$ we infer that $G = N$. By the principle of extensionality it follows that $h_x = h_x'$.

Now let H be the subset of N consisting of those elements x for which an operation h_x exists. Taking h_0 to be the identity operation on N (such that $h_0 y = y$ for all $y \in N$), we see that $h_0 0 = 0$ (thus satisfying 3.1), and for any $y \in N$, $h_0(Sy) = Sy = S(h_0 y)$ (thus satisfying 3.2). Hence $0 \in H$. Furthermore, H is closed under S. For suppose $x \in H$, so that an operation h_x exists. Let h_{Sx} be the operation on N such that $h_{Sx} y = S(h_x y)$ for all $y \in N$. We see that $h_{Sx} 0 = S(h_x 0) = Sx$ (thus satisfying 3.1 for Sx), while for any $y \in N$, $h_{Sx}(Sy) = S(h_x(Sy)) = S(S(h_x y)) = S(h_{Sx} y)$ (thus satisfying 3.2 for Sx); whence $Sx \in H$. Applying Axiom P3, which holds for $\langle N, 0, x\rangle$ by assumption, we infer that $H = N$, which proves the lemma.

Using this lemma we can complete the proof of Theorem II by exactly the same argument used earlier to infer the existence of an addition operation from Theorem I. The reader will notice that in this earlier argument of Section 2 we used Theorem I to obtain the statement of our present lemma; as noted there, the remainder of the argument was of a general set-theoretical character independent of the axioms P1–P3, and so can be applied to the induction model $\langle N, 0, S\rangle$ of Theorem II.

When we consider multiplication we find a situation entirely analogous to the one we have just encountered for addition. By a multiplication operation for an induction model $\langle N, 0, S\rangle$ we mean a binary operation $\cdot$ on N such that the equations

$$11.1 \qquad x\cdot 0 = 0,$$

$$11.2 \qquad x\cdot(Sy) = (x\cdot y) + x,$$

hold for all $x, y\in N$, where $+$ is the addition operation for the model. Employing a proof closely patterned after the one given above for Theorem II, we can show that in every induction model there is one and only one multiplication operation. We leave the reader to supply the details of such a proof.

4. Operations in Peano models obtained by primitive recursion. When we come to the operation of exponentiation the situation changes. By an exponential operation for an induction model $\langle N, 0, S\rangle$ is meant a binary operation exp such that the equations

$$12.1 \qquad x \exp 0 = S0,$$

$$12.2 \qquad x \exp (Sy) = (x \exp y)\cdot x,$$

hold for all $x, y\in N$, where $\cdot$ is the multiplication of the model. This time there is no analogue of Theorem II, for it is simply not true that every induction model has an exponential operation; a counter-example is provided by the model $\langle N'', a_0, T\rangle$ which we have considered in Section 2, where a_0 and a_1 are distinct objects, $N'' = \{a_0, a_1\}$, and $Ta_0 = a_1$, $Ta_1 = a_0$. It is a simple matter to show that if exp were a binary operation on N'' satisfying 12.1 and 12.2 for all $x, y\in N''$, then we would have $a_0 \exp z = a_0$ (for any $z\in N''$) from 12.2 (since $w\cdot a_0 = a_0$ for all $w\in N''$), while from 12.1 we get $a_0 \exp a_0 = a_1$. This contradiction shows that $\langle N'', a_0, T\rangle$ possesses no exponential operation.

However, by an application of Theorem I we can show that every *Peano* model possesses a unique exponential operation. In fact, we can get a general result of which this is a special case.

THEOREM III. *Let $\langle N, 0, S\rangle$ be any Peano model, let f be a unary operation on N, and let g be a ternary operation on N. Then there exists one and only one binary operation j on N such that the equations*

$$13.1 \qquad jx0 = fx,$$

$$13.2 \qquad jx(Sy) = gxy(jxy),$$

hold for all $x, y\in N$.†

† This function j is said to be obtained by *primitive recursion* from f and g. More generally, one allows for the possibility of obtaining an n-ary operation j from an $(n-1)$-ary operation f and $(n+1)$-ary operation g, for any $n = 1, 2, \cdots$; we have selected the case $n = 2$ merely to simplify notation, since the idea of the proof is the same for any n.

(If we take the particular case where f is the unary operation on N such that $fx=S0$ for all $x\in N$, and where g is the ternary operation on N such that $gxyz=z\cdot x$ for all x, y, z, then clearly the function j obtained will be the exponential operation.)

Proof. As remarked in the course of our proof of Theorem II, it is sufficient to show that for each $x\in N$ there is a unique unary operation j_x on N such that

$$14.1 \qquad i_x0 = fx,$$

$$14.2 \qquad i_x(Sy) = gxy(j_xy),$$

for all $y\in N$; for then we can infer Theorem III by a general set-theoretical argument which holds for all models (*i.e.*, which does not involve any of the Axioms P1–P3). To this end, let N_1 be the set $N\times N$ of all ordered pairs $\langle y, z\rangle$ for all $y, z\in N$. For each $x\in N$ let 0_x be the element $\langle 0, fx\rangle$ of N_1, and let S_x be the unary operation on N_1 such that $S_x\langle yz\rangle=\langle Sy, gxyz\rangle$ for all $y, z\in N$. Applying Theorem I to the Peano model $\langle N, 0, S\rangle$ and the model $\langle N_1, 0_x, S_x\rangle$, we see that for each $x\in N$ there is a unique mapping h_x of N into N_1 such that the equations

$$15.1 \qquad h_x0 = 0_x,$$

$$15.2 \qquad h_x(Sy) = S_x(h_xy),$$

hold for all $y\in N$.

Next let L and R be the mappings of N_1 into N such that $L\langle x, y\rangle=x$ and $R\langle x, y\rangle=y$ for all $x, y\in N$, and for each $x\in N$ let j_x and k_x be the unary operations on N such that $j_xy=R(h_xy)$ and $k_xy=L(h_xy)$ for all $y\in N$. We shall show that j_x satisfies 14.1 and 14.2 for all $y\in N$, but first it is necessary to prove that $k_xy=y$ for all $y\in N$. To do this, let G be the subset of N consisting of those elements y such that $k_xy=y$. Since $k_x0=L(h_x0)=L0_x=0$, we see that $0\in G$. Next, let y be any element of G. Then $k_x(Sy)=L(h_x(Sy))=L(S_x(h_xy))$ by 15.2. By definition of S_x we have

$$\begin{aligned} S_x(h_xy) &= \langle S(L(h_xy)), gx(L(h_xy))(R(h_xy))\rangle \\ &= \langle S(k_xy), gx(k_xy)(j_xy)\rangle, \end{aligned}$$

so that $k_x(Sy)=S(k_xy)$. Since $y\in G$ it follows that $k_x(Sy)=Sy$, whence $Sy\in G$. We have thus shown that G is closed under S, and since Axiom P3 holds for $\langle N, 0, S\rangle$ we conclude that $G=N$, *i.e.*, that $k_xy=y$ for all $y\in N$. Returning to the formula

$$S_x(h_xy) = \langle S(k_xy), gx(k_xy)(j_xy)\rangle,$$

derived above, we see that it can now be simplified to

$$S_x(h_xy) = \langle Sy, gxy(j_xy)\rangle.$$

Since $j_x(Sy)=R(h_x(Sy))=R(S_x(h_xy))$, we get at once that $j_x(Sy)=gxy(j_xy)$,

showing that j_x satisfies 14.2. On the other hand $j_x0 = R(h_x0) = R0_x = fx$, so that j_x also satisfies 14.1. To complete the proof of Theorem III it merely remains to show that j_x is the *only* unary operation on N which satisfies these two equations; this can be done along the same lines of argument which we used when we inferred from Theorem I that there was only one addition operation for any Peano model—as we leave the reader to verify.

5. The relation between Peano models and induction models. Why is it that the operations of addition and multiplication exist in every induction model, while the existence of an exponential operation can be guaranteed only for Peano models? To answer this, we must first understand the relation which holds between Peano models and more general induction models. It turns out that from the algebraic viewpoint this relation is a very simple and natural one.

Let $\mathfrak{N} = \langle N, 0, S\rangle$ and $\mathfrak{N}_1 = \langle N_1, 0_1, S_1\rangle$ be any two models. We say that $\mathfrak{N}_1$ is a *homomorphic image* of $\mathfrak{N}$ if there exists a homomorphism h of $\mathfrak{N}$ *onto* $\mathfrak{N}_1$, *i.e.*, a function h whose domain is N, and whose range is the whole of N_1, such that $h0 = 0_1$ and $h(Sx) = S_1(hx)$ for all $x \in N$.

THEOREM IV. *Let $\mathfrak{N} = \langle N, 0, S\rangle$ be a Peano model and $\mathfrak{N}_1 = \langle N(, 0_1, S_1\rangle$ an arbitrary model. A necessary and sufficient condition that $\mathfrak{N}_1$ be a homomorphic image of $\mathfrak{N}$ is that $\mathfrak{N}_1$ be an induction model.*

Proof of necessity. Suppose that $\mathfrak{N}_1$ is a homomorphic image of $\mathfrak{N}$, and let h be a homomorphism of $\mathfrak{N}$ onto $\mathfrak{N}_1$. Let G_1 be any subset of N_1 such that $0_1 \in G_1$ and such that G_1 is closed under S_1. To show that $\mathfrak{N}_1$ is an induction model, it suffices to show that $G_1 = N_1$. To this end we consider the subset G of N consisting of just those elements x such that $hx \in G_1$. Since $h0 = 0_1$ we see that $0 \in G$. Furthermore, G is closed under S. For suppose $x \in G$, so that $hx \in G_1$. Now $S_1(hx) \in G_1$ since G_1 is closed under S_1. But $h(Sx) = S_1(hx)$ since h is a homomorphism. Thus $h(Sx) \in G_1$ and so $Sx \in G$; hence G is closed under S. Since $\langle N, 0, S\rangle$ satisfies Axiom P3, we see that $G = N$. This means that $hx \in G_1$ for all $x \in N$. Since h has domain N and range N_1 we infer that $G_1 = N_1$.

Proof of sufficiency. Suppose that $\langle N_1, 0_1, S_1\rangle$ is an induction model. Since $\langle N, 0, S\rangle$ is a Peano model we can apply Theorem I to infer the existence of a (unique) homomorphism h of $\mathfrak{N}$ into $\mathfrak{N}_1$. To complete the proof of our theorem it is only necessary to show that the range of h is the whole of N_1. But clearly the range of h contains 0_1, since $h0 = 0_1$, and it is closed under S_1 since for any element z in the range of h there must be an $x \in N$ for which $hx = z$, whence $S_1z = S_1(hx) = h(Sx)$ so that S_1z is also in the range of h. But $\mathfrak{N}_1$ satisfies Axiom P3, so the range of h is N_1 as was to be shown.

From Theorem IV there follows an important and well-known corollary.

THEOREM V. *Any two Peano models are isomorphic.*†

Proof. Let $\mathfrak{N}$ and $\mathfrak{N}_1$ be any Peano models. By Theorem IV there is a homo-

† That is, there is a homomorphism of one onto the other which is one-one.

morphism h of $\mathfrak{N}$ onto $\mathfrak{N}_1$ and a homomorphism h_1 of $\mathfrak{N}_1$ onto $\mathfrak{N}$. Clearly the composed function (h_1h), which is a unary operation on N such that $(h_1h)x = h_1(hx)$ for all $x \in N$, is a homomorphism of $\mathfrak{N}$ into $\mathfrak{N}$. But by Theorem I there is only one homomorphism of $\mathfrak{N}$ into $\mathfrak{N}$, and obviously the identity operation on $\mathfrak{N}$ is such a homomorphism. Hence (h_1h) is the identity operation: $(h_1h)x = x$ for all $x \in N$. It easily follows that h is one-one, and hence h is an isomorphism of $\mathfrak{N}$ onto $\mathfrak{N}_1$.

The principal significance of Theorem V is a metamathematical consequence to the effect that for a certain large set of sentences containing the symbols "N", "0", and "S", any sentence of the set which is true of one Peano model is also true of every other Peano model; hence if a sentence of this set is true of some particular Peano model such as the system $\langle N^*, 0^*, S^* \rangle$ of natural numbers as we know them intuitively, it will be a logical consequence of the Axioms P1–P3. The set of sentences to which this metamathematical result applies contains all those sentences about models which have any interest for us in the present work. Accordingly, from now on, instead of speaking of an arbitrary Peano model we may speak of the system $\langle N^*, 0^*, S^* \rangle$ of natural numbers.

In Theorem II we have seen that every induction model $\mathfrak{N} = \langle N, 0, S \rangle$ possesses a unique operation of addition, $+$. Now that we see from Theorem IV that $\mathfrak{N}$ bears a close relation to the system of natural numbers $\mathfrak{N}^* = \langle N^*, 0^*, S^* \rangle$, it is natural to inquire into the relation between the operation $+$ of $\mathfrak{N}$ and the addition operation $+^*$ of the system $\mathfrak{N}^*$. The answer to this inquiry is given by the following result.

THEOREM VI. *Let $\mathfrak{N} = \langle N, 0, S \rangle$ be any induction model and $+$ its operation of addition. Let h be the unique homomorphism of $\mathfrak{N}^*$ onto $\mathfrak{N}$. Then $h(x +^* y) = hx + hy$ for all $x, y \in N^*$.*

Proof. Let x be any element of N^*, and let G be the subset of N^* such that $y \in G$ if and only if $h(x +^* y) = hx + hy$. Since $h(x +^* 0^*) = hx = hx + 0 = hx + h0^*$ we see that $0^* \in G$. Now let y be any element of G. Then $h(x +^* y) = hx + hy$, so that $h(x +^* S^*y) = h(S^*(x +^* y)) = S(h(x +^* y)) = S(hx + hy) = hx + S(hy) = hx + h(S^*y)$, and hence $S^*y \in G$. Since G is closed under S^* we apply Axiom P3 (which holds for $\mathfrak{N}^*$) to conclude that $G = N^*$; and this proves the theorem.

In the terminology of set-theory and algebra we express the content of Theorem VI by saying that the operation $+$ on N is the image (under the homomorphism h) of the operation $+^*$ on N^*. The corresponding theorem for multiplication is equally true, and can be proved in essentially the same way. Now there is a well-known result for general algebraic systems to the effect that if an equation is satisfied identically in one system, the same equation will be satisfied in any homomorphic image of the system if each operation of the original system which enters into the equation is replaced by an operation (of the second system) which is its image under the homomorphism. It follows that such identities as the associative, commutative, and distributive laws which hold for $+^*$ and

$\cdot^*$ in the system $\mathfrak{N}^*$ will also hold for the operations $+$ and $\cdot$ of any induction model. Actually this fact can be shown directly, without appeal to any result which holds for general algebraic systems; for upon inspecting the usual derivation of these laws for Peano models, proceeding from 1.1, 1.2, 11.1, and 11.2, it will be found that only Axiom P3, never P1 or P2, is used.

6. Congruence relations. Another fact about homomorphisms which is well known for general algebraic systems, and which can be established directly in a simple manner for the models under investigation here, is their close connection with congruence relations. By a *congruence relation* for a model $\mathfrak{N}=\langle N, 0, S\rangle$ is meant an equivalence relation R on N† such that whenever xRy then also $(Sx)R(Sy)$. If h is any homomorphism of $\mathfrak{N}$ into some other model, then the relation R_h such that for any x, $y\in N$ we have xR_hy if and only if $hx=hy$, is a congruence relation. Conversely, for every congruence relation R of $\mathfrak{N}$ there is a homomorphism h of $\mathfrak{N}$ onto some other model $\mathfrak{N}_R$, such that $R_h=R$. To construct $\mathfrak{N}_R$ we take N_R to be the class of all equivalence sets x_R for all $x\in N$, we let S_R be the operation on N_R (whose existence follows from the fact that R is a congruence relation of $\mathfrak{N}$) such that $S_Rx_R=(Sx)_R$ for all $x\in N$, and we set $\mathfrak{N}_R=\langle N_R, 0_R, S_R\rangle$. If we define h to be the function mapping N onto N_R such that $hx=x_R$ for all $x\in N$, we easily see that h is a homomorphism of $\mathfrak{N}$ onto $\mathfrak{N}_R$ and that $R_h=R$.

If h_1 and h_2 are homomorphisms of $\mathfrak{N}$ onto models $\mathfrak{N}_1$ and $\mathfrak{N}_2$ respectively, and if $R_{h_1}=R_{h_2}$, then $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are isomorphic. It follows that every homomorphic image of a model $\mathfrak{N}$ is isomorphic to one of the models $\mathfrak{N}_R$ determined (in the manner described above) by some congruence relation R of $\mathfrak{N}$. In view of Theorems IV and V we therefore see that every induction model is isomorphic to a model $\mathfrak{N}_R^*$ determined by some congruence relation R of the system $\mathfrak{N}^*$ of natural numbers.

As it happens, we can give an explicit description of all congruence relations on $\mathfrak{N}^*$ in terms of the familiar ordering relation $<$ on N^*.‡ Namely, let m, n be any elements of N^*. We define the relation $R_{m,n}$ on N^* by the rule that $xR_{m,n}y$ if and only if one of the following two conditions holds: (i) x, $y<n$ and $x=y$, (ii) x, $y\geqq n$ and for some $z\in N^*$ either $x=y+^*(z\cdot^* m)$ or $y=x+^*(z\cdot^* m)$.

THEOREM VII. *A binary relation R on N^* is a congruence relation of $\mathfrak{N}^*$ if and only if it is the identity relation on N^* or there exist numbers m, $n\in N^*$ such that $R=R_{m,n}$.*

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† An equivalence relation on N is a binary relation R, having N both for its domain and range, which is reflexive, symmetric, and transitive. It is an elementary fact of set-theory that for each such relation R there is a partition of N into disjoint subsets such that two elements are in the same subset if and only if these elements are in the relation R. The subset containing an element x is called the *equivalence set of x under R*; we will denote it x_R.

‡ This ordering relation can be introduced into the axiomatic theory of $\mathfrak{N}^*$ (*i.e.*, into the theory of Peano models), by the definition $x<y$ if and only if there is an element $z\neq 0^*$ such that $x+^*z=y$. Since every induction model possesses an addition function, we can use this definition to define a

We shall leave the proof of this theorem to the reader, contenting ourselves with giving the following hint. If R is a congruence relation other than the relation of identity, there must be an $x \in N^*$ such that xRy for some $y \neq x$. Choose n to be the least of these numbers x. It follows from this choice of n that there are numbers $z \neq 0$ such that $nRn +^* z$; choose m to be the least of these numbers z. It can then be shown that $R = R_{m,n}$.

The congruence relations $R_{m,0}$ are the so-called modular congruences which have been extensively studied in the theory of numbers.§ The induction model $\mathfrak{N}^*_{R_{m,0}}$ corresponding to such a relation is simply the system of residue classes modulo m, and the operations of addition and multiplication of this model are the familiar $+ (\text{mod } m)$ and $\cdot \ (\text{mod } m)$. The congruence relations $R_{m,n}$ for $n \neq 0$ do not seem to have received much attention in the literature,# but a little reflection will suffice to give the reader a clear intuitive picture of the models $\mathfrak{N}^*_{R_{m,n}}$, as well as of the corresponding operations of addition and multiplication, in this case also.

Incidentally, it is evident that if R is one of the modular congruence relations $R_{m,0}$, then S^*_R is a permutation of the elements of N^*_R so that $\mathfrak{N}^*_R$ satisfies Axiom P2 in this case. On the other hand, if R is one of the congruence relations $R_{m,n}$ for $n \neq 0$, it is clear that $0^*_R \neq S^*_R x$ for all $x \in N_R$, so that $\mathfrak{N}^*_R$ satisfies Axiom P1 in this case. Thus every induction model satisfies either P1 or P2, as mentioned in Section 1.

Let f be any operation on $\mathfrak{N}^*$—let us say a binary operation for definiteness. If R is an arbitrary congruence relation on $\mathfrak{N}^*$, there is in general no binary operation g on N^*_R which is a homomorphic image of f under the homomorphism h which corresponds to R. It is not hard to see that the necessary and sufficient condition for the existence of such an image operation g is that for all $x, x_1, y, y_1 \in N^*$ such that xRx_1 and yRy_1 we have $(fxy)R(fx_1y_1)$. If this condition holds then g is the operation on N^*_R such that $gx_Ry_R = (fxy)_R$ for all $x, y \in N^*$.

For example, although $2 \equiv 2 \pmod 3$ and $0 \equiv 3 \pmod 3$ we have $2^0 \not\equiv 2^3 \pmod 3$. Hence the exponential operation of N^* has no homomorphic image in $N^*_{R_{3,0}}$. On the other hand, $+^*$ and $\cdot^*$ are examples of what we may call *universal operations* on N^*; that is, they are operations f with the property that for *any* congruence relation R of N^*, $(fxy)R(fx_1y_1)$ whenever x, y, x_1, y_1 are elements of N^* such that xRx_1 and yRy_1.† It is for this reason that every induction model possesses operations of addition and multiplication.

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relation $<$ in every *induction* model; but in general the relation so obtained will not be an ordering relation.

§ It is customary, in works on number-theory, to write $x \equiv y \pmod m$ instead of $xR_{m,0}y$.

A brief reference may be found in a note by H. S. Vandiver, Bull. Amer. Math. Soc. vol. 40, 1934, pp. 914–920.

† We may call f a *modular operation* if it has this property for all *modular* congruence relations R (and not necessarily for the other congruence relations). An interesting characterization of the modular operations has been given by N. G. de Bruijn. (Cf. Proc. Kon. Ned. Ak. Wetensch. Amsterdam, series A, 58 (Indagationes Math. 17), 1955, pp. 363–367).

Suppose that an operation j on N^* is obtained by primitive recursion from operations f and g which are universal; is j necessarily universal? The example of the exponential operation shows that this is not the case. However, if j happens to be commutative then it *will* be universal: this can be shown by a straightforward generalization of the proof of Theorem II. Thus it is because the exponential operation on N^* is noncommutative that we cannot extend the proof of Theorem II to show the existence of an exponential operation in every induction model.

Of course the condition of commutativity, although sufficient to guarantee universality under the stated conditions, is by no means necessary. For example, the operation j such that $jxy = x^2 \cdot y$ for all $x, y \in N^*$ is a universal operation, and it is obtained by primitive recursion from the universal functions f, g such that $fx = 0$ and $gxyz = z + x^2$ for all $x, y, z \in N^*$; but j is noncommutative.

7. A characterization of Peano models. In Section 2 we have seen that the justification for a definition by mathematical induction in a model $\mathfrak{N}$ is the existence of a unique homomorphism of $\mathfrak{N}$ into some other model; and thus Theorem I constitutes a justification of *all* definitions by mathematical induction in Peano models. As it happens, this property is characteristic for Peano models: these are the *only* models in which all definitions by mathematical induction are justified.

THEOREM VIII. *Let $\mathfrak{N}$ be a model such that, for any model $\mathfrak{N}_1$ there is a unique homomorphism h of $\mathfrak{N}$ into $\mathfrak{N}_1$. Then $\mathfrak{N}$ is a Peano model.*

Proof. Let $\mathfrak{N} = \langle N, 0, S \rangle$, and suppose that $\mathfrak{N}$ satisfies the hypothesis of the theorem. We shall show first that $\mathfrak{N}$ satisfies Axiom P3. To this end let G be any subset of N which contains 0 and is closed under S. Let H be the complement of G (with respect to N), and assume that H is not empty. Let k be a one-one mapping of H onto a set P which is disjoint from N. Let M be the union of N and P. Define a unary operation T on M, as follows: If $x \in N$ then $Tx = Sx$; if $x \in H$ and $Sx \in H$ then $T(kx) = k(Sx)$; if $x \in H$ and $Sx \in G$ then $T(kx) = Sx$. Let $\mathfrak{M}$ be the model $\langle M, 0, T \rangle$. It is clear that the mapping h_1 of N into M such that $h_1 x = x$ for all $x \in N$ is a homomorphism of $\mathfrak{N}$ into $\mathfrak{M}$. On the other hand, consider the following mapping h_2 of N into M: If $x \in G$ then $h_2 x = x$; if $x \in H$ then $h_2 x = kx$. It is not difficult to see that h_2 is also a homomorphism of $\mathfrak{N}$ into $\mathfrak{M}$, and that it is distinct from h_1. But this contradicts the hypothesis of our theorem, and hence shows that it was incorrect to assume H nonempty. H is empty, and so $G = N$. That is, $\mathfrak{N}$ must be an induction model.

From Theorem IV we can now infer that there is a homomorphism h' of $\mathfrak{N}^*$ onto $\mathfrak{N}$. On the other hand the hypothesis of our theorem assures us that there is a homomorphism h of $\mathfrak{N}$ into $\mathfrak{N}^*$. As in the proof of Theorem V we consider the composed function (hh') which is a homomorphism of $\mathfrak{N}^*$ into itself, and so (by Theorem I) must be the identity operation on $\mathfrak{N}^*$. It follows that h' is one-one, and hence is an isomorphism of $\mathfrak{N}^*$ onto $\mathfrak{N}$. This proves the theorem.

It is clear from Theorem VIII that any proof of Theorem I must employ all of the Axioms P1–P3, as claimed in Section 2.

8. Conclusion. We have now finished our discussion of the concept of definition by mathematical induction in the theory of numbers, based upon Peano's axiomatic foundation for this theory. In the literature of mathematics there are various other types of inductive definitions employed, *e.g.*, definition by transfinite induction, or by induction in certain types of partially ordered systems. Many of the ideas of this paper can be generalized to cover these other types of induction.

While the method of presenting the material in this paper may have some claim to originality, several of the proofs which were given are well known to mathematicians. In particular, this is true of the two proofs (one given in detail, the other outlined) of Theorem I. According to Professor Alonzo Church, the origin of the first proof goes back to the Hungarian mathematician L. Kalmár, while the idea of the second proof should be credited to P. Lorenzen, and D. Hilbert and P. Bernays, who discovered the proof independently and published their work nearly simultaneously.† The proof of Theorem II is given by E. Landau‡ who credits it to L. Kalmár. However, Landau fails to note the significance of the fact that the proof does not use Axioms P1 and P2.§

† Kalmár's article appears in Acta Sci. Math. (Szeged), vol. 9, No. 4, 1950, pp. 227–232. Lorenzen's work appears in Monatsh. Math. und Phys., vol. 47, 1938–39, pp. 356–358, and the proof of Hilbert and Bernays appears in the Appendix to Volume 2 of their book Grundlagen der Mathematik.

‡ See E. Landau, Foundations of Analysis.

§ (Added October 2, 1958.) There has just come to my attention an article with several ideas closely related to those of this paper: H. Lenz, Zur Axiomatik der Zahlen, Acta Math. Acad. Sci. Hungar., vol. IX, 1958, pp. 33–44.

GLOBAL EQUILIBRIUM THEORY OF CHARGES ON A CIRCLE*

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1. Introduction. Recently electronic calculations were made [1] to see if there is some "brute force" solution to an old problem of finding equilibrium positions of charged particles constrained to lie on a sphere and acted upon by mutual Newtonian repulsion (see Föppl [2]). While some new stable minima seemed plausible on the basis of the calculation, the problem of verifying unstable equilibrium positions was too formidable for conclusive results from numerical data at present.

Surprisingly enough the analogous two-dimensional unstable equilibrium problem is far from trivial and yet can be analyzed completely; but it never seems to have made its way into the literature [6]. Here we would consider n

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point charges, not necessarily equal, constrained to lie on a circle and acting under a fairly general law of repulsion which permits particles to coincide for "closure." We find all critical configurations, stable and otherwise.

The conditions are then perfect for the application of the Poincaré-Morse [3] global theory. Here we calculate s, the number of negative squares of the quadratic form that gives the potential near each critical configuration. These values of s are interrelated by a formula ((4.3) below) of the "Euler characteristic" type.

As intuition suggests, there is only one stable critical configuration (excluding rigid rotation) for every ordering of noncoincident charges (for example the obvious regular n-gon for equal charges). The Poincaré-Morse theory reduces to identities among partition "tallies" that are not trivial in nature.

2. Symbols. We describe each of the n positive charges q_i by an angular variable θ_i, $0 \leqq i \leqq n-1$, and use a potential function

$$V(\theta_0, \cdots, \theta_{n-1}) = \sum_{i>j} V_{ij}, \tag{2.1}$$

where we introduce the function of "distance" $f(\phi)$ and

$$V_{ij} = q_i q_j f(\theta_i - \theta_j). \tag{2.2}$$

Here $f(\phi)$ is subject to the properties that $f(\phi)$ is even, $f(\phi)$ has period 2π, $f''(\phi) > 0$, and $f'(\epsilon)/\epsilon \to -\infty$ as $\epsilon \to 0$ (clearly satisfied by the logarithmic potential, $f(\phi) = -\log \left| 2 \sin \tfrac{1}{2}\phi \right|$). Under these conditions $f(0)$ may be infinite, making a *global* theory inapplicable as the set of angles θ_i would not be permitted to range over a *closed* manifold. We, therefore, always modify $f(\phi)$ to a new function $f_\epsilon(\phi)$ defined for ϵ which is always taken "as small as the context requires":

$$\begin{aligned} f_\epsilon(\phi) &= f(\phi), \qquad \epsilon \leqq \left|\phi\right| \leqq \pi; \\ f_\epsilon(\phi) &= f(\epsilon) + f'(\epsilon)(\phi^2 - \epsilon^2)/2\epsilon, \qquad 0 \leqq \left|\phi\right| \leqq \epsilon; \end{aligned} \tag{2.3}$$

so as to preserve continuity and differentiability. We modify V according to this new function $f_\epsilon(\phi)$, and drop the subscript ϵ for convenience. If ϵ is chosen small enough none of the nonsingular configurations ($\theta_i \neq \theta_j$ when $i \neq j$), are affected locally, while $V(\theta_0, \cdots, \theta_{\epsilon-1})$ is continuously differentiable on a closed manifold.

To see the closed manifold more clearly, we take advantage of the circular symmetry by setting $\theta_0 = 0$ and

$$V(0, \theta_1, \cdots, \theta_{n-1}) = W(\theta_1, \cdots, \theta_{n-1}). \tag{2.4}$$

Then W is defined on a closed manifold $\mathfrak{M}$ consisting of the $(n-1)$-dimensional torus or the product of the $n-1$ intervals modulo 2π

$$0 \leqq \theta_i < 2\pi, \qquad i = 1, \cdots, n-1. \tag{2.5}$$

The *critical* points (of stability, metastability, etc.), are all characterized by

the conditions at $\theta_i=\theta_i^{(0)}$,

$$\partial W/\partial\theta_i = 0, \qquad i = 1, \cdots, n-1. \tag{2.6}$$

Each critical point of $\mathfrak{M}$ corresponds to a configuration of $n-1$ values of $\theta_i^{(0)}$ and $\theta_0^{(0)}(\equiv 0)$. These configurations form k clusters of m_t coincident values of $\theta_i^{(0)}$, $0\leqq i\leqq n-1$,

$$1 \leqq k \leqq n, \qquad 1 \leqq m_t \leqq n \ (t = 1, \cdots, k), \qquad \sum_{t=1}^{k} m_t = n. \tag{2.7}$$

We define m_t-1 as the degree of *degeneracy of each cluster* and

$$D = \sum_{t=1}^{k} (m_t - 1) = n - k \tag{2.8}$$

as the degree of *degeneracy of the configuration* $\theta_i^{(0)}$.

3. Positive and negative eigenvalues. The potential energy W in the neighborhood of a critical point is denoted by a quadratic form in $\alpha_i=\theta_i-\theta_i^{(0)}$, $0\leqq i\leqq n-1$, so that, $\alpha_0\equiv 0$ as before. The basic lemma used here concerns the form

$$Q^*(\alpha_i) = \sum_{i>j}^{*} \frac{1}{2} p_{ij}(\alpha_i - \alpha_j)^2, \tag{3.1}$$

where the * superscript refers to summation over a previously designated subset of the indices $\{0, 1, \cdots, n-1\}$, which may or may not include 0.

LEMMA 3.1. *If there are m indices present, and $p_{ij}>0$, the form $Q^*(\alpha_i)$ has $m-1$ positive eigenvalues.*

Proof. For convenience we can take the case where the subscript 0 is included in the set of m indices

$$Q^*(\alpha_i) = \sum_{i>j}^{*} \alpha_i\alpha_j P_{ij}, \qquad (i \neq 0, j \neq 0);$$

$$P_{ij} = -p_{ij} \ (i \neq j), \qquad P_{ii} = \sum_{j}^{*} p_{ij} + p_{i0} \qquad (j \neq i, i \neq 0, j \neq 0); \tag{3.2}$$

so that for the $m-1$ indices i (excluding "0," of course),

$$P_{ii} > \sum_{j}^{*} |P_{ij}| \qquad (j \neq i, j \neq 0). \tag{3.3}$$

This inequality is sufficient for the positiveness of eigenvalues by a well-known theorem [4]. To use this lemma we need two further results:

LEMMA 3.2. *If we consider the family of quadratic forms $Q(\alpha_i) = Q_1(\alpha_i) + \eta Q_2(\alpha_i)$, then as $\eta\to\infty$ the form $Q(\alpha_i)$ will have at least as many positive (negative) eigenvalues as $Q_2(\alpha_i)$.*

LEMMA 3.3. *If we project the Euclidean space α_i onto a subspace of lower dimension by imposing linear conditions on the α_i, then we obtain no greater number of positive (negative) eigenvalues in the form $Q(\alpha_i)$ after projection than before projection.*

These lemmas are very simple to prove, and naturally are quite classical [5]. Generally, $W(\theta_i) - W(\theta_i^{(0)})$ is approximated by

$$(3.4) \qquad Q(\alpha_i) = \sum_{i>j} f''(\theta_i^{(0)} - \theta_j^{(0)}) \tfrac{1}{2} q_i q_j (\alpha_i - \alpha_j)^2, \qquad (\alpha_i = \theta_i - \theta_i^{(0)}).$$

For the coincident points ($\theta_i^{(0)} = \theta_j^{(0)}$), of the configuration, from (2.3), $f''(\theta_i^{(0)} - \theta_j^{(0)}) = f'(\epsilon)/\epsilon$, which is infinitely negative compared with the derivatives that occur when $\theta_i^{(0)} \neq \theta_j^{(0)}$, as $\epsilon \to 0$.

Thus asymptotically, as $\epsilon \to 0$, $Q(\alpha_i)$ becomes the sum of k forms of type (3.1), (but with negative coefficients), formed by dropping the terms of (3.4) in which i and j belong to different clusters. By Lemma (3.1), the form $Q^*(\alpha)$ has exactly $\sum(m_i - 1) = D$ *negative* eigenvalues, and by Lemma (3.2), $Q(\alpha_i)$ has at least as many.

On the other hand, if we make all θ_i equal for those subscripts i belonging to each one of the k clusters, we can apply the argument of the preceding paragraph directly and see the existence of $k-1$ *positive* eigenvalues. Since $D+k-1 = n-1$, we have accounted for all eigenvalues in (3.4), on the assumption that ϵ be sufficiently small, depending on the charges q_i. Thus the number s of negative eigenvalues is D, the degeneracy, *i.e.*,

$$(3.5) \qquad D = n - k = s.$$

4. Global theory. The critical points are next classified according to the number s of negative eigenvalues in the quadratic form. We saw $s = D$ in this problem.

LEMMA 4.1. *To every ordered partition of subscripts $\{0, 1, \cdots, n-1\}$ into $k(\leqq n)$ clusters (each cluster being unordered), a unique critical configuration is determined, for ϵ small enough.*

We shall prove this theorem by induction on k. First of all it is clear that at least one such critical configuration is determined since we can identify all m points of each cluster and have k charges, each of value $\sum_{q_i}$ (summed over each cluster). Then W is a continuous function of $k-1$ variables (exception being made of the fixed zero cluster) and must achieve a minimum.

To see that the minimum is unique we use induction. For instance, when $k=2$ the fact that $f''(\phi) > 0$ for $|\phi| < \epsilon$ is sufficient. Let us assume the truth of Lemma 4.1 for all values of k up to n (inclusive). We then see that the number of critical points M_s corresponding to s negative eigenvalues is exactly the number of partitions of n points into k clusters keeping one fixed. Thus

$$M_s = \sum (n-1)!/(m_1-1)!m_2!\cdots m_k!, \qquad n-1 \geqq s \geqq 0, \tag{4.1}$$

subject to $k=n-s$, and

$$\sum_{i=1}^{k} m_i = n, \qquad m_i \geqq 1. \tag{4.2}$$

The global theory calls for the relation [3]

$$-\sum_{s=0}^{n-1} (-)^s M_s = \chi \tag{4.3}$$

for χ the Euler characteristic of $\mathfrak{M}$. But in our case $\chi=0$, and relation (4.3) is a combinatorial identity that can be proved by induction on n. (This is left to the reader.) It should be noted that the identity is *not* equivalent to the triviality,

$$\chi = -(1-1)^{(n-1)}. \tag{4.4}$$

In fact, $M_s \geqq \binom{n-1}{s}$, with equality only for $s=n-1$.

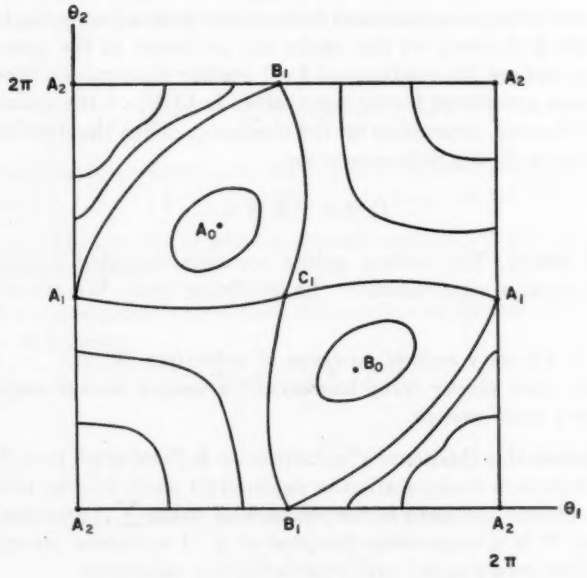

FIG. 1

We are now in a position to resume the proof of Lemma 4.1. If we replace n by $n+1$, the only value of s for which (4.1) does not apply is $s=0$, $k=n+1$. Then M_0 would be *greater* than the summation if some partition failed to produce

a unique critical point. But by (4.3), independently verified, M_0 is determined by the other M_s and the uniqueness could not fail for $k=n+1$.

5. Illustration. (Fig. 1.) We now take $n=3$ and sketch the level lines for $W(\theta_1, \theta_2)$. We note that the range of θ_i determines a square of side 2π with opposite sides identified to form the conventional "torus." For convenience we take equal charges at $\theta_0(\equiv 0)$, θ_1, and θ_2. We find the partitions of $n=3$ with the critical points as noted.

MINIMA $\{A_0, B_0\}$

$3 = 1 + 1 + 1;$ $\quad k = 3,\ s = 0;$ $\quad M_0 = 2!/0!1!1! = 2.$

SADDLE POINTS $\{A_1, B_1, C_1\}$

$3 = 2 + 1;$ $\quad k = 2,\ s = 1;$ $\quad M_1 = 2!/1!1!$

$3 = 1 + 2;$ $\quad k = 2,\ s = 1;$ $\quad + 2!/0!2! = 3.$

MAXIMUM $\{A_2\}$

$3 = 3;$ $\quad k = 1,\ s = 2;$ $\quad M_2 = 2!/2! = 1.$

6. Final conclusion. If we go back to the stage where no coincident charges could occur, we find the following result:

THEOREM. *Let n positive point charges q_i be placed at angles θ_i on a circle and be subject to a potential of mutual repulsion*

$$V = \sum_{i>j} f(\theta_i - \theta_j) q_i q_j,$$

for which $f(\phi) = f(-\phi)$, $f(\phi + 2\pi) = f(\phi)$, $f''(\phi) > 0$ and $f'(\epsilon)/\epsilon \to -\infty$ as $\epsilon \to 0$. Then ignoring rotations, there is exactly one configuration for which V is a relative minimum, and in which the θ_i occur in a preassigned order clockwise around the circle.

The proof is immediate if we use Lemma 4.1. A more direct proof might be difficult to find.

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ON BERNOULLI PLAY WITH LIMITED RESOURCES

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Let $\{x_k\}$ be a sequence of mutually independent Bernoulli random variables. Suppose that at each trial a nonnegative quantity of a limited resource is forfeited; say σ_1, if the trial outcome is zero, and $\sigma_2 > \sigma_1$, if the outcome is 1. Every play will then terminate after a finite number of trials, N.

Introducing the concept of a "stop" function we shall derive simple exact expressions for the probabilities

$$
\begin{aligned}
P(j) &= \Pr\{x_1 + \cdots + x_N = j\}, \\
Q(N) &= \Pr\{\text{number of trials in a play} = N\},
\end{aligned} \tag{1}
$$

under the assumption that trials are continued until depletion of the resource. This is a typical compound distribution problem ([2], p. 221), however, the classical formula is not applicable since N and j here are correlated random variables; a large j is usually accompanied by a small N.*

Construct a "stop" function, $s(n, j)$, for $n \geqq j$, which indicates the remaining amount of the resource after the nth trial, the cumulative number of successes being j. After the nth trial no further trials are permitted if $s(n, j) \leqq 0$, otherwise at least one more trial is allowed. The function $s(n, j)$ is such that

$$s(n+1, j) < s(n, j), \tag{2}$$

$$s(n, j+1) < s(n, j). \tag{3}$$

Let $f = \max j$ for the set $\{j \mid s(j, j) > 0\}$; then a play cannot terminate with fewer than $(f+1)$ trials, and $(f+1)$ is the maximum number of attainable successes. For each j, $0 \leqq j < f+1$, determine integer N_j defined by

$$s(N_j, j) > 0, \qquad s(N_j + 1, j) \leqq 0. \tag{4}$$

It follows that $N_{j-1} \geqq N_j$. Also, no one play of the game may have more trials than (N_0+1). Define $N_{f+1} = f$, since for every $n \geqq (f+1)$, $s(n, f+1) \leqq 0$; also define $N_{-1} = N_0$.

In what follows we shall use the function G (with $q = 1 - p$) defined by

$$
\begin{aligned}
G(N, j, p) &= p^j \sum_{c=0}^{N-j} \binom{j+c-1}{j-1} q^c = \sum_{i=j}^{N} \binom{N}{i} p^i q^{N-i}, \qquad j \leqq N, \\
&= 0, \qquad j > N.
\end{aligned}
$$

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* The author's first thoughts on this type of problem in the form of application to missile firing doctrines were presented in A Comparison of Firing Doctrines for a Sequentially Fired Weapon *vs.* a Non-Saturation Attack (U), George W. Morgenthaler, Report 58-1, Institute for Air Weapons Research, University of Chicago (Air Force Contract No. 33(616)-3274, Weapons Guidance Lab., Wright Air Development Center).

The latter expression is tabled ([1], p. 251, or [4]), and the equality of these two expressions for G is well known ([3], p. 40).

THEOREM 1. *For* $j=0, 1, \cdots, (f+1)$

$$P(j) = \binom{N_j}{j} p^j q^{N_j-j+1} + G(N_{j-1}+1, j, p) - G(N_j, j, p), \tag{5}$$

where p *is the probability of success and* $q=1-p$.

Proof. The first term corresponds to the probability of j successes in N_j trials, with the one additional permissible trial resulting in failure. Conditions (3) and (4) guarantee that in order to attain j successes, at least (N_j+1) trials will occur.

If $(j-i)$ successes occur in the first N_j trials, the remaining i successes must occur by trial $(N_{j-1}+1)$. If not, at that or any trial thereafter at which a total of $(j-1)$ successes have occurred, trials would cease according to the stop rule and the jth success would never occur. Thus from trial (N_j+1) to trial $(N_{j-1}+1)$ the play would cease whenever the jth success occurs. The probability that the jth success occurs at the (N_j+1+c)th trial, $0 \leqq c \leqq (N_{j-1}-N_j)$ is given by the Pascal distribution ([2], p. 218), as

$$\binom{N_j+c}{j-1} p^j q^{N_j+1-j+c}.$$

The sum of these probabilities is easily shown to be

$$[G(N_{j-1}+1, j, p) - G(N_j, j, p)],$$

upon writing down the sum and transforming the variable of summation from c to, say, $k=(N_j+1-j+c)$.

The distribution of N is most easily obtained by writing down the expressions $P(j)$ and then setting $Q(N)$ equal to the sum of all terms among the expressions $P(j)$ for which the exponents of p and q sum to N. However, explicit formulae for $Q(N)$ can also be written.

THEOREM 2. *Given the sequence* $N_{f+1}+1 \leqq N_f+1 \leqq \cdots \leqq N_j+1 \leqq N_{j-1}+1 \leqq \cdots \leqq N_0+1$, *the distribution of* N, $f+1 \leqq N \leqq N_0+1$, *is given by*

$$Q(N) = \begin{cases} \binom{N-1}{j-1} p^j q^{N-j}, & \text{if } j,\ 0 < j \leqq f+1, \text{ is an integer such that} \\ & N_j + 1 < N < N_{j-1} + 1. \\ \sum_{i=0}^{k} \binom{N}{j-i} p^{j-i} q^{N-j+i} + \binom{N-1}{j} p^{j+1} q^{N-1-j}, & \text{if } j,\ 0 \leqq j \leqq f+1, \\ & \text{and } k,\ k \leqq j, \text{ are the largest integers such that} \\ & N = N_j + 1 = N_{j-1} + 1 = \cdots = N_{j-k} + 1. \end{cases} \tag{6}$$

Proof. When j successes are obtained there will not be fewer than (N_j+1)

trials, nor more than $(N_{j-1}+1)$. Thus to each integer j we associate the integers of the interval $I_j=[N_j+1, N_{j-1}+1]$. Then $I_{j+1}\cap I_j=N_j+1$; $I_j\cap I_{j-1}=(N_{j-1}+1)$. If N is such that for some j, $(N_j+1)<N<(N_{j-1}+1)$, then $N\in I_j$ only. Then N trials can occur only if the jth success occurs on the Nth trial. This probability is given in the first line of (6).

Alternatively, N may be of the form (N_j+1) for certain consecutive j's. If j and k are as stated in the theorem, N can occur when $(j-i)$ successes occur in N_{j-i} trials and the remaining trial is a failure, or when $(j-i-1)$ successes occur in N_{j-i} trials, the remaining trial being a success. The sum of the probabilities of these two events is

$$\binom{N_{j-i}+1}{j-i}p^{j-i}q^{N_j-i-j+i+1}, \qquad i=0, 1, \cdots, k,$$

and we may replace $(N_{j-i}+1)$ by N.

The only other way in which $N=N_j+1$ could occur is if the $(j+1)$st success was obtained on trial (N_j+1). This probability is the final term in the second line for $Q(N)$.

As a first example of such processes, consider a machine on a production line which produces one unit every T seconds. Suppose that imperfect or broken units occur with probability p at each trial and suppose that T_s additional seconds must then be allotted to eject the defective unit from the machine (possibly machine clean-up or temporary shutdown is necessary). The distribution of stoppages, and hence the expected number of seconds of shutdown per production day, may be calculated using (5). The stop function would be

$$s(n, j) = (\text{Total sec. per production day}) - nT - jT_s.$$

Secondly, if a particle of given initial kinetic energy is followed in a random walk in which more energy is lost for a step to the right than for movement to the left, (6) will give the distribution of the number of steps up to standstill.

In some processes it is of interest to invoke a second "stop" rule which terminates trials when a total of $B<(f+1)$ successes have occurred. Then (5) is valid only for $j<B$.

THEOREM 3. *Under the additional "stop" rule which terminates trials after B successes,*

$$P(B) = G(N_{B-1}+1, B, p). \tag{7}$$

Proof. The probability of the Bth success occurring at the $(B+i)$th trial is given by $\binom{B+i-1}{i}p^Bq^i$. If the Bth success has not occurred by trial $(N_{B-1}+1)$, there will not be sufficient of the resource remaining to achieve B successes. Summing these probabilities results in (7).

THEOREM 4. *Under the additional "stop" rule which terminates trials after B successes, the distribution of N, $B \leqq N \leqq N_0+1$, is given by*

$$Q(N) = \begin{cases} \binom{N-1}{B-1} p^B q^{N-B}, & \text{for } N \leqq N_{B-1}, \\ \text{same as (6)}, & \text{for } N > N_{B-1}+1, \end{cases} \tag{8}$$

$$Q(N_{B-1}+1) = 1 - \sum_{N \neq (N_{B-1}+1)} Q(N).$$

Proof. If $N > (N_{B-1}+1)$, then fewer than B successes occurred in the N trials and the probabilities of such N are treated in (6). If $N \leqq N_{B-1}$, necessarily the Bth success occurred on the Nth trial, thereby halting the trials according to the stop rule. Thus, by the Pascal distribution, $Q(N)$ has the value given in the first line of (8). For $N = N_{B-1}+1$, some plays which would have had $(N_{B-1}+1)$ trials may have been stopped because B successes were attained. This is true, for example, when $N_{B-1} = N_B$.

A final example concerns a process which terminates after B successes. Consider a defensive weapon which is sequential-fire rather than salvo-fire. Suppose that the depth of the zone of effective fire is d. A line-abreast wave of B attackers travelling at speed V assaults the defensive position.

Suppose that the time required for each shot is T and the time to observe the result of fire is T_0. Denote the switching time, *i.e.*, the time required to reorient fire-control equipment to a new attacker, by T_s.

Of the many possible firing doctrines, consider one in which the first target is fired upon and the result is observed. If the target survives, a second shot is fired at it and the result observed, etc. When the target is destroyed, the second target is fired upon and the result observed, etc., until it too is destroyed. Fire is then switched to the third target, etc., until either all targets are destroyed or the attackers have reached the defensive position. Suppose that the attack is nonsaturating, *i.e.*, if but one shot were directed at each attacker, all attackers could be fired upon before they reached the defensive position. Under this fire-observe-switch doctrine the number of shots fired and the number of attackers destroyed are correlated random variables. Theorems 3 and 4 apply.

The limited resource here is time, and an appropriate stop function is

$$s(n, j) = (d/V) - n(T + T_0) - jT_s + T_0. \tag{9}$$

This particular military example has alternative treatments at this point. Stop function (9) together with (4) defines N_j as the integers

$$N_j < \frac{(d/V) - jT_s + T_0}{T + T_0} \leqq N_j + 1. \tag{10}$$

This permits firing (N_j+1) shots even though there is not a full T seconds of

firing time left for the last shot. This also requires that if the last shot fired is a successful shot there must be sufficient time for switching even though firing will cease. It is possible to change these assumptions and, *e.g.*, to define N_j using the greatest integer notation, "[ ]," *i.e.*,

$$N_j + 1 = \left[\frac{(d/V) - jT_s + T_0}{T + T_0}\right], \tag{11}$$

thus insuring sufficient time for all (N_j+1) shots. Each alternative rule of this sort corresponds to a slightly different game and the corresponding probability distributions are different.

If N_j is defined by (10) or by (11), and $T_s \leqq T+T_0$ (a reasonable engineering assumption) then $0 \leqq (N_{j-1} - N_j) \leqq 1$. Hence for $j<B$, (5) gives

$$P(j) = \begin{cases} \binom{N_j+1}{j} p^j q^{N_j+1-j}, & \text{if } (N_{j-1} - N_j) = 0, \\ \binom{N_j+1}{j} p^j q^{N_j+1-j} + \binom{N_j+1}{j-1} p^j q^{N_j+2-j}, & \text{if } N_{j-1} - N_j = 1, \end{cases} \tag{12}$$

and $P(B)$ is given by (7).

Consider the specific instance of B aircraft attacking a missile-defended target and suppose $d=52.5$ n.m., $T=64$ sec., $T_s=20$ sec., $T_0=11$ sec., $V=596$ n.m./hr., $B=3$. Then by (10), $N_0=4$, $N_1=4$, $N_2=3$, $N_3=3$. By (7), (8), and (12),

$$P(0) = q^5, \qquad P(1) = 5pq^4,$$

$$P(2) = \binom{4}{2} p^2q^3 + \binom{4}{1} p^2q^3, \qquad P(3) = \sum_{j=3}^{4} \binom{4}{j} p^j q^{4-j};$$

$$Q(3) = p^3, \qquad Q(4) = 3p^3q + 6p^2q^2, \qquad Q(5) = 5pq^4 + 4p^2q^3 + q^5.$$

On the other hand, if (11) is used, $N_0=3$, $N_1=3$, $N_2=2$, $N_3=2$. By (7), (8), and (12),

$$P(0) = q^4, \qquad P(1) = 4pq^3,$$

$$P(2) = \binom{3}{2} p^2q + \binom{3}{1} p^2q^2, \qquad P(3) = p^3;$$

$$Q(3) = p^3 + 3p^2q, \qquad Q(4) = 3p^2q^2 + 4pq^3 + q^4.$$

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ON THE ENUMERATION OF POLYGONS

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1. Introduction. Given n equally spaced points on a circle, one may pick a first vertex in n ways, a second vertex in $(n-1)$ ways, . . . , an nth vertex in 1 way, and return to the starting point in 1 way, for a total of $n!$ polygonal paths. Two polygonal paths which differ only in starting point or orientation will be called *identical polygons*. If, besides possible difference in starting point and orientation, two polygons differ only by a plane rotation, they will be termed *equivalent*. If, in addition to possible differences of these three types, two polygons differ only by a reflection through some axis, they will be called *similar*. Using a combinatorial formula of Pólya, it has been possible to obtain explicit expressions for the number of classes $E(n)$ of equivalent n-gons, and for the number of classes $S(n)$ of similar n-gons.

2. The formulas. Let n exceed 2. It is convenient to separate the even from the odd values of n. In all cases, summation is extended over the divisors d of n, and $\phi(a)$ is Euler's totient function.

$$(1) \qquad E_{\text{odd}}(n) = \frac{1}{2n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d\right)$$

$$(2) \qquad E_{\text{even}}(n) = \frac{1}{2n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{n/2}\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)!\right)$$

$$(3) \qquad S_{\text{odd}}(n) = \frac{1}{4n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{(n-1)/2}\cdot n^2\cdot\left(\frac{n-1}{2}\right)!\right)$$

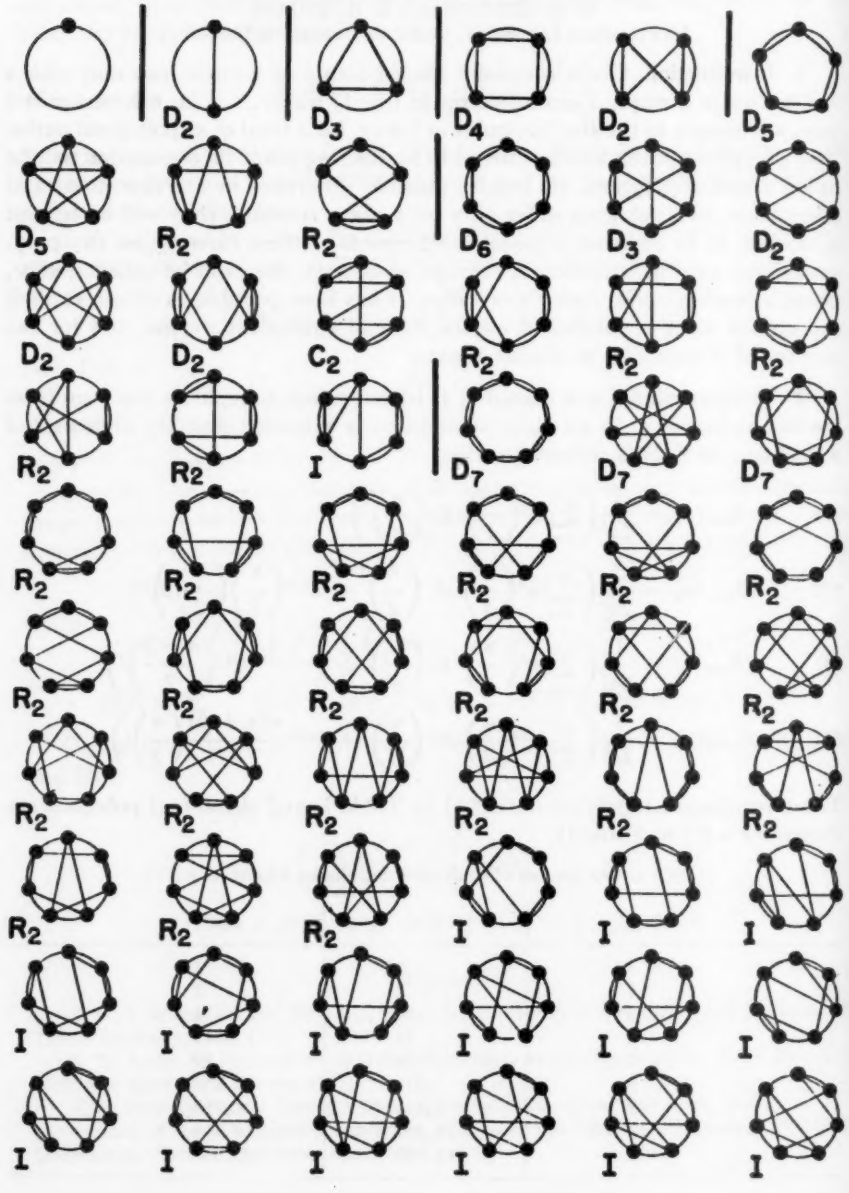
$$(4) \qquad S_{\text{even}}(n) = \frac{1}{4n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{n/2}\cdot\frac{n(n+6)}{4}\left(\frac{n}{2}\right)!\right).$$

These results are tabulated for $n \leqq 11$ in Table I, and the actual polygons are shown for $n \leqq 7$ in Table II.

TABLE I. The number of equivalent and similar n-gons for $n \leqq 11$.

n	$E(n)$	$S(n)$
3	1	1
4	2	2
5	4	4
6	14	12
7	54	39
8	332	202
9	2,246	1,219
10	18,264	9,468
11	164,950	83,435

TABLE II. All polygons with $\leqq 7$ sides, with their symmetry groups. The groups I and C_2 correspond to polygons with reflections which are not rotations.



3. Deriving the formulas. The basic combinatorial formula [1], [2] for the number C of equivalence classes established in a set S by the operation of a group G of order N is

$$C = \frac{1}{N} \sum_{g \in G} I(g), \tag{5}$$

where $I(g)$ is the number of objects s in S left fixed by the operator g in G, *i.e.* for which $g(s) = s$.

The original set S for present purposes is the set of all $n!$ polygonal paths. The "identity group" contains $2n$ elements, for $n > 2$, based on the n starting points and two orientations of path. Whenever e is the unity element of the group G in (5), $I(e) =$ total size of set S. For the "identity group," none of the polygonal paths remain invariant under operations other than the unity. Hence the trivial result

$$I(n) = \frac{1}{2n}(n!) = \frac{1}{2}(n-1)!. \tag{6}$$

The "equivalence group" is the direct product of the identity group with the group of cyclic permutations of the n-gon, and has $(2n)(n) = 2n^2$ elements. For the purposes of equation (5), only the n cycling operators contribute to the sum, provided that polygons which are changed into identical polygons are counted in $I(g)$. Cycling by an angle $\theta_k = 2\pi k/n$, $1 \leqq k \leqq n$, leaves one or more polygons fixed. The set of invariant polygons under this rotation clearly depends only on the greatest common divisor $(n, k) = d$. Thus

$$E(n) = \frac{1}{2n^2} \sum_{k=1}^{n} I(\theta_k) = \frac{1}{2n^2} \sum_{d \mid n} \phi\left(\frac{n}{d}\right) I(\theta_d), \tag{7}$$

where ϕ is Euler's function, and θ_d also depends on n.

To obtain $I(\theta_d)$, we must determine the number of n-gons left fixed by the rotation $2\pi d/n$. Unless $d = n/2$ (in the case of even n), all invariant n-gons for θ_d can be constructed as follows:

Number vertices consecutively from 1 to n, and distinguish residue classes modulo d. A first vertex is picked in any of n ways. The next vertex is picked arbitrarily, but *outside* the residue class of the first point—thus, in $(n - n/d)$ ways. The next vertex is picked from a new residue class, in $(n - 2n/d)$ ways, and this process continues until all d residue classes are represented once. This "first loop" of the polygon can thus be formed in

$$n\left(n - \frac{n}{d}\right)\left(n - \frac{2n}{d}\right) \cdots \left(n - \frac{(d-1)n}{d}\right)$$

$$= n^d \left(\frac{d-1}{d}\right)\left(\frac{d-2}{d}\right) \cdots \left(\frac{1}{d}\right) = \left(\frac{n}{d}\right)^d \cdot d!$$

ways. The first point on the "second loop" must belong to the same residue class as the first point on the "first loop," and can be chosen in $\phi(n/d)$ ways. The future history of this polygon is now completely determined by the pattern within each loop, and the transition from one loop to the next. This transition does not lead to an early termination of the polygon, since it was restricted to be one of the $\phi(n/d)$ nondegenerate possibilities. (See particularly the second through fifth hexagons in Table II.)

For the case of odd n, this completes the derivation of

$$E_{\text{odd}}(n) = \frac{1}{2n^2} \sum_{d|n} \phi^2\left(\frac{n}{d}\right) \cdot d! \cdot \left(\frac{n}{d}\right)^d. \tag{1}$$

For even n, there is also the possibility that a rotation by π will reproduce the polygon with its orientation reversed. This can happen as follows:

There are $n/2$ pairs of antipodal points. A diameter can thus be drawn in $n/2$ locations. A choice of one member from each antipodal pair can be made in $2^{n/2}$ ways. The first members can be arranged sequentially in $(n/2)!$ ways. This leads to $2^{n/2}(n/2)(n/2)!$ ways of drawing a polygon which inverts upon itself. (See especially the sixth hexagon in Table II.) Hence the formula

$$E_{\text{even}}(n) = \frac{1}{2n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right) \cdot d! \cdot \left(\frac{n}{d}\right)^d + 2^{n/2}\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)!\right). \tag{2}$$

The "similarity group" contains reflections as well as rotations, for a total of $4n^2$ operations. Hence the relations

$$S_{\text{odd}}(n) = \frac{1}{2} E_{\text{odd}}(n) + \frac{1}{4n^2} \cdot n \cdot I(R_2), \tag{8}$$

where R_2 is the operation of reflection in any one of the n axes of the polygon, and

$$S_{\text{even}}(n) = \frac{1}{2} E_{\text{even}}(n) + \frac{1}{4n^2}\left(\frac{n}{2} I(R_2') + \frac{n}{2} I(R_2'')\right), \tag{9}$$

where R_2' and R_2'' refer to reflection in the vertex diagonals and edge diagonals, respectively.

To compute $I(R_2)$, there are n ways to pick a starting point. With respect to the symmetry axis, there is one solitary vertex and $(n-1)/2$ pair of matched vertices. There are $2^{(n-1)/2}$ ways to select first members of the pairs; and $((n-1)/2)!$ ways to order these first members. Hence

$$I(R_2) = n \cdot 2^{(n-1)/2} \cdot \left(\frac{n-1}{2}\right)!. \tag{10}$$

Substituting (10) into (8) yields the expression

$$(3) \qquad S_{\text{odd}}(n) = \frac{1}{4n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{(n-1)/2}\cdot n^2\cdot\left(\frac{n-1}{2}\right)!\right).$$

By an analogous counting process,

$$(11) \qquad I(R_2') = n\cdot 2^{n/2}\cdot\left(\frac{n}{2}-1\right)!$$

and

$$(12) \qquad I(R_2'') = n\cdot 2^{n/2-1}\cdot\left(\frac{n}{2}\right)!$$

Equations (9), (11), (12), and (2) combine to yield

$$(4) \qquad S_{\text{even}}(n) = \frac{1}{4n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{n/2}\frac{n(n+6)}{4}\cdot\left(\frac{n}{2}\right)!\right).$$

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ON THE THEORY OF LÖBELL ON TRANSFORMATION OF SURFACES

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1. Introduction. Frank Löbell [1] has given a discussion of the theory of transformation of surfaces. He pointed out that, for certain cases as the nature of the problem requires, the consideration of the mixed fundamental magnitudes of a pair of surfaces is opportune. But he did not give any example for this. In the present paper we propose to give such an example.

2. Parallel surfaces. A surface S' which is at a constant distance along the normal from another surface S is said to be parallel to S. As the constant distance may be chosen arbitrarily, the number of such parallel surfaces is infinite. Parallel surfaces have the property that the normals to the two surfaces at corresponding points are parallel [2]. This property agrees with the common requirement of some of the important formulas such as (12), (13), (15b), (15c), of Löbell. Therefore we may use these formulas to investigate parallel surfaces and so give our illustrative examples of Löbell's theory, as follows.

If $\mathbf{x}$ is the current point on the surface S, $\mathbf{c}$ the unit normal to that surface, and α the constant distance, the corresponding point on the parallel surface S' is

$$\mathbf{h} = \mathbf{x} + \alpha\mathbf{c}. \tag{1}$$

Let E, F, G and E', F', G' denote the fundamental magnitudes of the first order of S and S', respectively, and $\overline{E}, \overline{F}_1, \overline{F}_2, \overline{G}$, the mixed fundamental magnitudes of the first order of the surface pair $S \rightleftarrows S'$, that is, let

$$\begin{gathered} E = \mathbf{x}_u\cdot\mathbf{x}_u,\ F = \mathbf{x}_u\cdot\mathbf{x}_v,\ G = \mathbf{x}_v\cdot\mathbf{x}_v; \qquad E' = \mathbf{h}_u\cdot\mathbf{h}_u,\ F' = \mathbf{h}_u\cdot\mathbf{h}_v,\ G' = \mathbf{h}_v\cdot\mathbf{h}_v; \\ \overline{E} = \mathbf{x}_u\cdot\mathbf{h}_u,\ \overline{F}_1 = \mathbf{x}_u\cdot\mathbf{h}_v,\ \overline{F}_2 = \mathbf{x}_v\cdot\mathbf{h}_u,\ \overline{G} = \mathbf{x}_v\cdot\mathbf{h}_v. \end{gathered} \tag{2}$$

Then, by (1), we have

$$\begin{aligned} \overline{E} &= \mathbf{x}_u\cdot(\mathbf{x}_u + \alpha\mathbf{c}_u) = E - \alpha L, \quad & \overline{F}_1 &= \mathbf{x}_u\cdot(\mathbf{x}_v + \alpha\mathbf{c}_v) = F - \alpha M, \\ \overline{F}_2 &= \mathbf{x}_v\cdot(\mathbf{x}_u + \alpha\mathbf{c}_u) = F - \alpha M, \quad & \overline{G} &= \mathbf{x}_v\cdot(\mathbf{x}_v + \alpha\mathbf{c}_v) = G - \alpha N, \end{aligned} \tag{3}$$

where L, M, N are the fundamental magnitudes of the second order of S.

In the first place, we find from (3) that

$$\overline{E} - E = -\alpha L, \quad \overline{F}_1 - F = \overline{F}_2 - F = -\alpha M, \quad \overline{G} - G = -\alpha N. \tag{4}$$

It then follows that

$$\begin{aligned} \alpha^2(LN - M^2) &= (\overline{E} - E)(\overline{G} - G) - (\overline{F}_1 - F)(\overline{F}_2 - F) \\ &= (\overline{E}\overline{G} - \overline{F}_1\overline{F}_2) + (EG - F^2) - [\overline{E}G + E\overline{G} - F(\overline{F}_1 + \overline{F}_2)]. \end{aligned}$$

Dividing by $EG - F^2$, we obtain

$$\frac{\alpha^2(LN - M^2)}{EG - F^2} = \frac{\overline{E}\overline{G} - \overline{F}_1\overline{F}_2}{EG - F^2} - \frac{\overline{E}G + E\overline{G} - F(\overline{F}_1 + \overline{F}_2)}{EG - F^2} + 1. \tag{5}$$

The term on the left side of (5) is equal to α^2 times the Gaussian curvature of S and the first term on the right is the ratio of the Gaussian curvature of S to that of S'.* The second term on the right is equal to $(m_1+m_2)\cos\delta$ by [1], (15c), where m_1, m_2 are, respectively, the lengths of the orthogonal vectors $a\mathbf{h}_u+b\mathbf{h}_v$, $c\mathbf{h}_u+d\mathbf{h}_v$ on S' corresponding to the orthogonal unit vectors $a\mathbf{x}_u+b\mathbf{x}_v$, $c\mathbf{x}_u+d\mathbf{x}_v$ on S, and δ is the angle of rotation from $m_1(a\mathbf{x}_u+b\mathbf{x}_v)$ to $a\mathbf{h}_u+b\mathbf{h}_v$. In order to determine the value of the second term on the right side of (5), we note that in (3), $\overline{F}_1 = \overline{F}_2$. Therefore by [1], (15b), we have

$$W(m_1 + m_2)\sin\delta = 0, \tag{6}$$

where $W = \sqrt{(EG - F^2)}$. Since m_1+m_2 is, in general, different from zero (and the case $m_1+m_2=0$ is excluded throughout the paper), then $\delta = 0$ or π. Hence we obtain

* This was pointed out without proof by Löbell ([1], II). A proof may be given as follows: Write $W = \mathbf{c}\cdot(\mathbf{x}_u \times x_v)$, $W' = \mathbf{c}'\cdot(\mathbf{h}_u \times \mathbf{h}_v)$, where $\mathbf{c}'$ is the unit normal to S' at $\mathbf{h}(u, v)$, and $T^2 = LN - M^2$, $T'^2 = L'N' - M'^2$. Then we must prove that the first term on the right side of (5) is equal to $(T^2/W^2) \div (T'^2/W'^2)$. By a well-known formula of differential geometry, $(T^2/W)\mathbf{c} = \mathbf{c}_u \times \mathbf{c}_v = \mathbf{c}_u' \times \mathbf{c}_v' = (T'^2/W')\mathbf{c}' = (T'^2/W')\mathbf{c}$, so that $T^2/W = T'^2/W'$. The required result now follows from this and the fact that, by [1], (12), $(\overline{E}\overline{G} - \overline{F}_1\overline{F}_2)/(EG - F^2) = W'/W$.

$$\frac{\overline{E}G + E\overline{G} - F(\overline{F}_1 + \overline{F}_2)}{EG - F^2} = \pm (m_1 + m_2). \tag{7}$$

Such being the case, (5) takes the form

$$\alpha^2 K = (K/K') \mp (m_1 + m_2) + 1. \tag{8}$$

If the Gaussian curvature K of S is constant and equal to $1/a^2$, and we take $\alpha = \pm a$, we have

$$\pm (m_1 + m_2) = K/K'. \tag{9}$$

Thus we get the first property of parallel surfaces:

THEOREM 1. *With every surface S of constant Gaussian curvature $1/a^2$ there are associated two surfaces parallel to S and distant $\pm a$ from it, on which the sum of the moduli of the orthogonal tangential vectors corresponding to the orthogonal tangential unit vectors of S is equal to the ratio of the Gaussian curvature of S to that of the associated surface at corresponding points.*

Secondly, since* $K/K' = m_1 m_2$, it follows from (9) that

$$\pm \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = 1. \tag{10}$$

We then obtain the second property of parallel surfaces:

THEOREM 2. *With every surface S of constant Gaussian curvature $1/a^2$ there are associated two surfaces parallel to S and distant $\pm a$ from it, on which the sum of the reciprocals of the moduli of the orthogonal tangential vectors corresponding to the orthogonal tangential unit vectors of S is unity.*

Finally, it can be shown that the well-known theorems of Bonnet may be found in a similar way. Let the lines of curvature on S be taken as parametric curves, so that $F = M = 0$. Then if ρ_a, ρ_b denote the principal radii of curvature on S and K_a, K_b, the principal curvatures of S, we have

$$\mathbf{h}_u = \mathbf{x}_u + \alpha \mathbf{c}_u = [(\rho_a - \alpha)/\rho_a]\mathbf{x}_u, \mathbf{h}_v = \mathbf{x}_v + \alpha \mathbf{c}_v = [(\rho_b - \alpha)/\rho_b]\mathbf{x}_v.$$

Hence

$$\begin{aligned} \overline{E} &= \mathbf{x}_u \cdot \mathbf{h}_u = (1 - \alpha K_a)E, \quad \overline{F}_1 = \mathbf{x}_u \cdot \mathbf{h}_v = (1 - \alpha K_b)F = 0, \\ \overline{G} &= \mathbf{x}_v \cdot \mathbf{h}_v = (1 - \alpha K_b)G, \quad \overline{F}_2 = \mathbf{x}_v \cdot \mathbf{h}_u = (1 - \alpha K_a)F = 0. \end{aligned} \tag{11}$$

But from (7),

$$\pm (m_1 + m_2) = \frac{\overline{E}G + E\overline{G} - F(\overline{F}_1 + \overline{F}_2)}{EG - F^2} = \frac{\overline{E}G + E\overline{G}}{EG},$$

* Since $m_1(a\mathbf{x}_u + b\mathbf{x}_v) \times m_2(c\mathbf{x}_u + d\mathbf{x}_v) = (a\mathbf{h}_u + b\mathbf{h}_v) \times (c\mathbf{h}_u + d\mathbf{h}_v)$, we have $m_1 m_2 (ad - bc)\mathbf{x}_u \times \mathbf{x}_v = (ad - bc)\mathbf{h}_u \times \mathbf{h}_v$. But ([1], II) $(K/K')(\mathbf{x}_u \times \mathbf{x}_v) = \mathbf{h}_u \times \mathbf{h}_v$, so that $K/K' = m_1 m_2$.

which, by virtue of (11), may be written $\pm(m_1+m_2)=2-\alpha(K_a+K_b)$. On substituting this in (8), we have

$$\alpha^2K = (K/K') + \alpha J - 1, \tag{12}$$

where J denotes the mean curvature of S.

Now, if the mean curvature J is constant and equal to $1/a$ and we take $\alpha=a$, we find that $a^2K=K/K'$, *i.e.*, $K'=1/a^2=$const. Consequently we get the important theorem of Bonnet on parallel surfaces:

THEOREM 3. *With a surface of constant mean curvature* $1/a$ *there is associated a parallel surface of constant Gaussian curvature* $1/a^2$ *at a distance* a *from it.*

Again, for a pseudospherical surface of curvature $-1/a^2$, we may take $\alpha=\pm a$, so that $\alpha^2K=-1$. Then from (12), we have $K/K'=\mp J$, that is, $K'=\pm 1/(a^2J)$. From this we have:

THEOREM 4. *With a pseudospherical surface* S *of curvature* $-1/a^2$ *there are associated two surfaces parallel to* S *and distant* $\pm a$ *from it, of which the Gaussian curvature is equal to the reciprocal or its negative of* a^3J, *according as the distance is* $+a$ *or* $-a$, *where* J *is the mean curvature of* S.

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REMARKS ON THE INTRINSIC EQUATIONS OF TWISTED CURVES

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1. Introduction. The shape of a twisted curve is defined by two functions of its arc length s, the curvature $k_1(s)$ and the torsion $k_2(s)$. The equations $k_1=k_1(s)$, $k_2=k_2(s)$ are known as "intrinsic equations." Their solution would give us the Cartesian coordinates of the curve but for an arbitrary rigid motion.

In the case of a plane curve, $k_2(s)=0$, the solution reads

$$x(s) = \int \cos\left(\int k_1(s)ds\right)ds; \qquad y(s) = \int \sin\left(\int k_1(s)ds\right)ds. \tag{1}$$

There is not much known in the general case. The problem can be reduced to the solution of a certain Riccati differential equation. But this type can be solved in closed form only under exceptional conditions. The main significance of the method is to assure us of the existence of a solution. Even if the Riccati

equation is solved numerically, the procedure is extremely complicated. Several sets of particular integrals of the differential equation are needed to construct the coordinates of the requested twisted curve. This situation is unsatisfactory. It is the purpose of this paper to explore a purely geometrical approach which is easier to visualize.

The suggested procedure consists of several steps. First it will be proved that we can write down the coordinates of the requested curve immediately if we know the unit vector of the principal normal, $\mathbf{n}(s_1)$, as function of the arc length s_1 of the twisted curve.

The unit vector of the principal normal describes a spherical curve. We denote its arc length by s_2, its geodesic curvature by $k_g(s_2)$. The arc length s_2 and the geodesic curvature $k_g(s_2)$ can be expressed by the given curvature $k_1(s_1)$ and the given torsion $k_2(s_1)$ of the requested twisted curve. Therefore the problem of solving the two intrinsic equations of a general twisted curve has been reduced to the task of managing the single intrinsic equation of a spherical curve. This is the core of the new approach.

One would guess perhaps that it should be possible to solve the intrinsic equation of a spherical curve similarly to the treatment of a plane curve as in (1). The writer spent much time years ago, to do just this. It was in vain. Another detour seems to be unavoidable. It is possible to map any spherical curve into a plane curve of arc length s_3 and curvature $k_p(s_3)$ of such a kind that

$$\int k_g(s_2)ds_2 \leftrightarrow \int k_p(s_3)ds_3. \tag{2}$$

This fundamental correspondence gives us sufficient information to construct numerically the plane image of the spherical curve. It is possible to go back from the image to the original. Therefore we are able to find the unit vector $\mathbf{n}$ of the principal normal of the requested curve. The components of $\mathbf{n}$ are in numerical form. This is not a disadvantage since the remaining quadratures could not be done in closed form anyhow.

In the future we are going to use the original curvature $k_1(s_1)$ and the original torsion $k_2(s_1)$ as input, to integrate (1) the curve in the image plane numerically, to find automatically (2) the components of the unit vector $\mathbf{n}$ of the principal normal, and to construct finally (3) Cartesian coordinates of the requested twisted curve. Although this procedure seems to be a long detour, the various steps are easily visualized and convenient for programming. A general numerical solution of the intrinsic equations of twisted curves is in reach, probably for the first time.

2. Description of the vector of a twisted curve by the unit vector of its principal normal.* The vector of the requested twisted curve with respect to an

* Cf. the discussion in G. Scheffers, Anwendung der Differential- und Integralrechnung auf Geometrie, Bd. I, 1901, p. 244.

arbitrary Cartesian system of coordinates is denoted by $\mathbf{r}(s_1)$ where s_1 is the arc length of the curve. The unit vectors of the tangent, the principal normal, and the binormal are—in this order—$\mathbf{t}(s_1)$, $\mathbf{n}(s_1)$, $\mathbf{b}(s_1)$. We denote the first curvature by $k_1(s_1)$, the second curvature (*i.e.*, the torsion) by $k_2(s_1)$, the total curvature $[k_1^2(s_1)+k_2^2(s_1)]^{1/2}$ by $k(s_1)$. Frenet's formulae become, with these notations:

$$
\begin{aligned}
d\mathbf{t}(s_1)/ds_1 &= +k_1(s_1)\mathbf{n}(s_1), \\
d\mathbf{n}(s_1)/ds_1 &= -k_1(s_1)\mathbf{t}(s_1) - k_2(s_1)\mathbf{b}(s_1), \\
d\mathbf{b}(s_1)/ds_1 &= +k_2(s_1)\mathbf{n}(s_1).
\end{aligned} \tag{3}
$$

Let us form the cross product

$$[\mathbf{n} \times (d\mathbf{n}/ds_1)] = k_1\mathbf{b} - k_2\mathbf{t}. \tag{4}$$

It is possible to express the vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ by the three mutually orthogonal vectors $\mathbf{n}$, $d\mathbf{n}/ds_1$, $[\mathbf{n}\times(d\mathbf{n}/ds_1)]$. The results read:

$$
\begin{aligned}
\mathbf{t} &= -(1/k^2)(k_1(d\mathbf{n}/ds_1) + k_2[\mathbf{n} \times (d\mathbf{n}/ds_1)]), \\
\mathbf{n} &= \mathbf{n}, \\
\mathbf{b} &= -(1/k^2)(k_2(d\mathbf{n}/ds_1) - k_1[\mathbf{n} \times (d\mathbf{n}/ds_1)]).
\end{aligned} \tag{5}
$$

We need

$$d^2\mathbf{n}/ds_1^2 = -(dk_1/ds_1)\mathbf{t} - k^2\mathbf{n} - (dk_2/ds_1)\mathbf{b}. \tag{6}$$

It follows that

$$\mathbf{n}\cdot(d\mathbf{n}/ds_1)(d^2\mathbf{n}/ds_1^2) = (dk_1/ds_1)k_2 - (dk_2/ds_1)k_1 = k_2^2(d(k_1/k_2)/ds_1). \tag{7}$$

On the other hand,

$$(d\mathbf{n}/ds_1)^2 = k^2 = k_2^2(1 + (k_1/k_2)^2). \tag{8}$$

Division of (7) by (8) yields

$$
\begin{aligned}
k_1/k_2 &= \tan \alpha(s_1), \\
\alpha(s_1) &= \int [\{\mathbf{n}(d\mathbf{n}/ds_1)(d^2\mathbf{n}/ds_1^2)\}/(d\mathbf{n}/ds_1)^2]\, ds_1.
\end{aligned} \tag{9}
$$

This means that the angle α is a known function of the arc length s_1. It follows

$$k_1(s_1) = k(s_1)\sin \alpha(s_1), \qquad k_2(s_1) = k(s_1)\cos \alpha(s_1). \tag{10}$$

If we insert these expressions into (5.1), we get

$$\mathbf{t}(s_1) = -(1/k(s_1))(\sin \alpha(s_1)\cdot(d\mathbf{n}/ds_1) + \cos \alpha(s_1)[\mathbf{n} \times (d\mathbf{n}/ds_1)]). \tag{11}$$

Since

$$\mathbf{r}(s_1) = \int \mathbf{t}(s_1)ds_1, \tag{12}$$

the vector $\mathbf{r}(s_1)$ of the requested twisted curve is known if we can find the unit vector $\mathbf{n}(s_1)$ of the principal normal. Equation (12) completes our first step.

3. The geodesic curvature of the spherical curve formed by the unit vector of the principal normal. The unit vector $\mathbf{n}(s_1)$ of the principal normal of the requested twisted curve describes a spherical curve. From

$$k(s_1) = \left| d\mathbf{n}/ds_1 \right| = \left| d\mathbf{n}/ds_2 \right| \cdot \left| ds_2/ds_1 \right| = \left| 1 \right| \cdot \left| ds_2/ds_1 \right|$$

follows for the arc length s_2 of this spherical curve

$$s_2 = \int k(s_1)ds_1. \tag{13}$$

Whereas s_1 has the dimension of a length, the dimension of s_2 is a pure number, *i.e.*, s_2 is measured in radians.

Temporarily we need the unit vectors of the tangent, the principal normal, and the binormal of the spherical curve, which we denote—in this order—by $\mathbf{t}^*(s_2)$, $\mathbf{n}^*(s_2)$, $\mathbf{b}^*(s_2)$. The corresponding first curvature is called $k_1^*(s_2)$. It is our aim to express this curvature by curvature and torsion of the requested twisted curve. We find:

$$\mathbf{t}^*(s_2) = (d\mathbf{n}/ds_1)(ds_1/ds_2) = -(k_1/k)\mathbf{t} - (k_2/k)\mathbf{b},$$

$$d\mathbf{t}^*(s_2)/ds_2 = -(1/k)((d(k_1/k)/ds_1)\mathbf{t} + k\mathbf{n} + (d(k_2/k)/ds_1)\mathbf{b}).$$

By squaring $k_1^{*2} = 1 + (1/k^2)((d(k_1/k)/ds_1)^2 + (d(k_2/k)/ds_1)^2)$. Since for any spherical curve the geodesic curvature k_g is connected to the first curvature k_1^* by the equation

$$k_g^2 = k_1^{*2} - 1, \tag{14}$$

we get

$$k_g = (1/k(s_1))d\alpha(s_1)/ds_1. \tag{15}$$

Considering (13) we find

$$\int k_g ds_2 = \alpha(s_1). \tag{16}$$

We need the angle α as a function of the arc length s_2 of the spherical curve. Therefore we write

$$\alpha(s_2) = \alpha[s_1(s_2)], \tag{17}$$

where the connection between s_1 and s_2 is given by (13).

Formulas (15) and (16) show that the single intrinsic equation of the spherical curve, described by the unit vector of the principal normal, is linked simply with the two intrinsic equations of the requested twisted curve. All these results are elementary. Probably they have been gained many times before. But apparently they were not used for attacking the intrinsic equations of twisted curves.

4. A particular mapping of a spherical curve on a plane. Let us assume that our unit vector of the principal normal is given by

$$\mathbf{n}(s_2) = \{\cos\beta(s_2)\cos\lambda(s_2);\ \cos\beta(s_2)\sin\lambda(s_2);\ \sin\beta(s_2)\}, \tag{18}$$

where s_2 is the arc length of this spherical curve. The coordinates (λ, β) form an auxiliary system. We are free in the choice of the location of the pole $\beta=\frac{1}{2}\pi$. Therefore we can assume that the equation $\beta(s_2)=0$ holds for $s_2=0$.

If $\omega(s_2)$ denotes the angle between spherical curve and meridian, we have

$$d\beta/ds_2 = \cos\omega; \qquad d\lambda/ds_2 = \sin\omega/\cos\beta. \tag{18}$$

The area between the curve from $s_2=0$ to an arbitrary s_2, the meridian $\lambda=\lambda(s_2)$, and the equator $\beta=0$ may be called $A(s_2)$. Then

$$A(s_2) = \int_{s_2=0}^{s_2} \sin\beta d\lambda. \tag{19}$$

The geodesic curvature of the spherical curve is $k_g(s_2)$ as before.

Now we consider a plane curve, expressed in polar coordinates (r, θ). The arc length is called s_3. The tangent forms with the straight line $\theta=0$ the angle $\gamma(s_3)$. The curvature is written $k_p(s_3)=d\gamma/ds_3$. It follows that

$$dr/ds_3 = \cos(\theta-\gamma); \qquad -d\theta/ds_3 = (1/r)\sin(\theta-\gamma). \tag{20}$$

The second members of (18) and (20) have the same formal structure. They become identical for the "translation"

$$\omega \leftrightarrow (\theta-\gamma); \qquad \cos\beta \leftrightarrow r. \tag{21}$$

From the second relationship we derive

$$-\sin\beta d\beta \leftrightarrow dr \quad \text{or} \quad d\beta/ds_2 \leftrightarrow -(1-r^2)^{-1/2}(dr/ds_2) \leftrightarrow dr/ds_3.$$

This leads us to

$$ds_2 \leftrightarrow -(1-r^2)^{-1/2}ds_3. \tag{22}$$

The correspondence of (18.2) and (20.2) yields

$$d\lambda/ds_2 \leftrightarrow -d\theta/ds_3 \quad \text{or} \quad d\theta \leftrightarrow -d\lambda(ds_3/ds_2) = \sin\beta d\lambda.$$

Therefore

$$A(s_2) = \int_{s_2=0}^{s_2} \sin \beta d\lambda \leftrightarrow \theta. \tag{23}$$

Whereas (21.2) gives us the radius vector r of the plane curve, the argument θ is defined by (23).

It remains to "translate" the geodesic curvature $k_g(s_2)$ into the plane curvature $k_p(s_3)$. It is known that $k_g(s_2)$ can be expressed by β and its first two derivatives. We have

$$k_g(s_2) = (d^2\beta/ds_2^2)[1 - (d\beta/ds_2)^2]^{-1/2} + \tan \beta[1 - (d\beta/ds_2)^2]^{1/2}. \tag{24}$$

According to (18.1), $d\beta/ds_2 = \cos \omega$. Thus

$$k_g(s_2) = -(d\omega/ds_2) + \tan \beta \sin \omega = d(A - \omega)/ds_2. \tag{25}$$

Now we are able to translate

$$k_g(s_2) \leftrightarrow (1 - r^2)^{1/2}(d\theta/ds_3) - (1 - r^2)^{1/2}(d\gamma/ds_3) + (1/r)(1 - r^2)^{1/2} \sin (\theta - \gamma).$$

According to (20.2) the first and third terms cancel each other. Further we have $d\gamma/ds_3 = k_p(s_3)$. Therefore

$$k_g(s_2) \leftrightarrow -(1 - r^2)^{1/2} k_p(s_3). \tag{26}$$

Finally we get, by multiplication with ds_2 according to (22),

$$\int k_g(s_2)ds_2 \leftrightarrow \int k_p(s_3)ds_3 \quad \text{or} \quad \alpha \leftrightarrow \gamma. \tag{27}$$

This basic correspondence was mentioned in the introduction. The angle $\alpha(s_1)$ occurred for the first time in (9). It was defined by $\alpha(s_1) = \arctan [k_1(s_1)/k_2(s_1)]$. It is an elementary function of the given curvature and the given torsion of the requested twisted curve. The same angle α measures the curvature of our plane image curve. This fact proves the close connection between the plane curve and the requested twisted trajectory. Our approach, which may look like a long detour, has brought us near to our goal.

Before we take the last step, let us summarize the results of this section in the "dictionary" given on the following page.

Although we are interested in twisted curves in general, it may be mentioned that this dictionary is a sharp tool for studying spherical curves. For instance, we may look for those spherical curves which correspond to a particular family of plane paths. Straight lines in the plane are images of great circles on the sphere. Circles of radius $R = R_0$ in the plane correspond to spherical curves $k_g(s_2) = -(1/R_0) \sin \beta$. This equation defines a class of random walks on the surface of the sphere.*

* H. von Schelling, Most frequent particle paths on the unit sphere, Trans. Amer. Geophys. Union, vol. 33, 1952, pp. 570–572.

SPHERE	PLANE
$\cos \beta$	r
$A = \int_{s_2=0}^{s_2} \sin \beta d\lambda$	θ
ds_2	$-(1 - r^2)^{-1/2} ds_3$
$k_g(s_2)$	$-(1 - r^2)^{1/2} k_p(s_3)$
$\alpha = \int k_g(s_2) ds_2 = (A - \omega)$	$\gamma = \int k_p(s_3) ds_3$
ω	$\theta - \gamma$
$d\beta/ds_2 = \cos \omega$	$dr/ds_3 = \cos(\theta - \gamma)$
$d\lambda/ds_2 = \sin \omega / \cos \beta$	$-d\theta/ds_3 = (1/r) \sin(\theta - \gamma)$

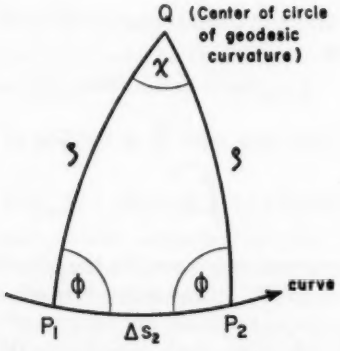

FIG. 1. Characteristic spherical triangle.

The connection between curvature k_1^*, torsion k_2^*, and geodesic curvature k_g of a spherical curve is shown in Figure 1 which depicts the characteristic spherical triangle. Here P_1P_2 is a short arc of the spherical curve of length Δs_2, approximated by a great circle. The triangle P_1P_2Q has equal angles ϕ at P_1 and P_2 and equal opposite sides $\rho = P_1Q = P_2Q$. If we assume that

$$\phi = \tfrac{1}{2}\pi - \tfrac{1}{2} k_g \Delta s_2, \tag{28}$$

we find

$$\angle P_1QP_2 = \chi = k_1^* \Delta s_2, \tag{29}$$

$$\cot \rho = k_g. \tag{30}$$

The point Q is the center of the circle of geodesic curvature. Curvature and tor-

sion can be expressed by the arc ρ also. The results read:

$$k_1^* = 1/\sin\rho, \qquad k_2^* = -d\rho/ds_2, \qquad k_g = \cot\rho. \tag{31}$$

Indeed a spherical curve is defined by its geodesic curvature alone. But sometimes it is useful to look at a spherical curve from the embedding three-dimensional Euclidean space. A knowledge of (31) is valuable for this purpose.

5. Application of the mapping. From our dictionary we know that $\gamma(s_3) = \alpha(s_1)$. We try to integrate numerically the plane curve. Its linear element is ds_3. On the other hand the angle α is given as function of s_1 by (9). In addition, differentiations and integrations in (11) and (12) refer to the argument s_1. Therefore it would be an advantage to use equally spaced values of s_1 from the start of numerical operations. We tabulate $\alpha(s_1)$ for $s_1 - S = 0, 1, \cdots$, where S is a positive constant. This means $\Delta s_1 \equiv 1$. Since

$$\Delta s_3 = -(1 - r^2)^{1/2}\Delta s_2 = -(1 - r^2)^{1/2}k(s_1)\Delta s_1,$$

we have

$$\Delta s_3 = -k(s_1)(1 - r^2)^{1/2}.$$

Let us start in the plane with an arbitrary point inside the unit circle $(x_0; y_0; r_0 = (x_0^2 + y_0^2)^{1/2} < 1)$, corresponding to $s_1 = S$. We write

$$\alpha_n = \alpha(S + n). \tag{32}$$

It follows that

$$\begin{gathered}\Delta x_n = -\cos\alpha_n k_n(s_1)(1 - r_{n-1}^2)^{1/2}; \qquad \Delta y_n = -\sin\alpha_n k_n(s_1)(1 - r_{n-1}^2)^{1/2}, \\ \Delta r_n = (1/r_{n-1})(x_{n-1}\Delta x_n + y_{n-1}\Delta y_n).\end{gathered} \tag{33}$$

This procedure is started with $n = 1$. All terms are known in this case.

The result of this numerical operation is a list of values of $r = \cos\beta$ for equally spaced arguments s_1. From $\cos^2\beta(d\lambda/ds_2)^2 = 1 - (d\beta/ds_2)^2$ we derive

$$\frac{d\lambda}{ds_1} = \frac{k}{\cos\beta}\left[1 - \frac{1}{k^2(1 - \cos^2\beta)}\left(\frac{d\cos\beta}{ds_1}\right)^2\right]^{1/2}. \tag{34}$$

An integration yields the longitude λ for equally spaced arguments s_1 also. This means that we know the vector $\mathbf{n}$, defined by (18), not as function of s_2 however, but of function of s_1. A simple numerical differentiation yields $d\mathbf{n}(s_1)/ds_1$. A following algebraic operation gives us $[\mathbf{n}(s_1) \times (d\mathbf{n}(s_1)/ds_1)]$.

The functions $k(s_1)$ and $\alpha(s_1)$ are known according to (8) and (9). Therefore we are able to tabulate the unit vector of the tangent of the requested twisted curve as function of s_1 according to (11). The final integration (12) gives us Cartesian coordinates of the twisted curve itself, correct but for arbitrary rigid motions of the system of coordinates. This completes our last step.

MATHEMATICAL NOTES

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A NOTE ON A CONVOLUTION

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Consider the integral transform $f(s)$ defined by

$$f(s) = \mathfrak{D}\{F(x)\} = s\int_1^\infty x^{-s-1}F(x)dx,$$

which converges for Re $s>\alpha$ if $F(x)=O(x^\alpha)$ and $F(x)$ is sectionally continuous. This is, of course, closely related to the Laplace transform since

$$f(s) = \mathfrak{D}\{F(x)\} = s\mathfrak{L}\{F(e^t)\}.$$

If we consider the transform $s^{-1}f(s)g(s)$ we have the convolution relation

$$s^{-1}f(s)g(s) = \mathfrak{D}\left\{\int_1^x F(x/u)G(u)u^{-1}du\right\} = \mathfrak{D}\left\{\int_1^x F(u)G(x/u)u^{-1}du\right\},$$

and the special case

$$s^{-1}f(s) = \mathfrak{D}\left\{\int_1^x u^{-1}F(u)du\right\}.$$

Another important relation is

$$\sum_{n=1}^\infty a_n n^{-s} = \mathfrak{D}\left\{\sum_{n\le x} a_n\right\}$$

in which case the integral is convergent for Re $s>\max(0, \sigma)$ where σ is the abscissa of convergence of the Dirichlet series.

Now since

$$\sum_{n=1}^\infty a_n n^{-s}\sum_{n=1}^\infty b_n n^{-s} = \sum_{n=1}^\infty\left(\sum_{d|n} a_d b_{n/d}\right)n^{-s},$$

we have from the convolution relation

$$\mathfrak{D}\left\{\int_1^x\left(\sum_{n\le u} a_n\right)\left(\sum_{n\le x/u} b_n\right)u^{-1}du\right\} = \mathfrak{D}\left\{\int_1^x\sum_{n\le u}\left(\sum_{d|n} a_d b_{n/d}\right)u^{-1}du\right\},$$

so that

$$I = \int_1^x\left(\sum_{n\le u} a_n\right)\left(\sum_{n\le x/u} b_n\right)u^{-1}du = \int_1^x\sum_{n\le u}\left(\sum_{d|n} a_d b_{n/d}\right)u^{-1}du.$$

These integrals can be converted into various forms involving sums,

$$I = \sum_{n\le x} a_n \int_n^x \Big(\sum_{k\le x/u} b_k\Big) u^{-1}du = \sum_{n\le x} a_n \int_1^{x/n} \Big(\sum_{k\le u} b_k\Big) u^{-1}du$$
$$= \sum_{n\le x}\sum_{k\le x/n} a_n b_k \log(x/kn) = \sum_{n\le x} \log(x/n) \sum_{d\mid n} a_d b_{n/d},$$

in any of which the a's and b's may be interchanged.

As an example consider $a_n = n^k$, $b_n = 1$, then

$$\int_1^x \Big(\sum_{n\le u} n^k\Big)\Big[\frac{x}{u}\Big]\frac{du}{u} = \int_1^x \Big(\sum_{n\le u} \sigma_k(n)\Big) u^{-1}du = \sum_{n\le x} \log(x/n)$$

in which $[x]$ denotes the greatest integer not exceeding x and $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n.

For a second example to illustrate relations between interesting integrals and sums, let $a_n = n$, $b_n = \mu(n)$, the Möbius function; then

$$\int_1^x \sum_{n\le x/u} \mu(n)\frac{[u][u+1]}{2u}du = \int_1^x \sum_{n\le u} \phi(n)u^{-1}du$$

in which $\phi(n)$ is Euler's ϕ-function.

A thorough discussion of the convolution (Faltung) for the Laplace transformation is given by G. Doetsch, *Handbuch der Laplace-Transformation*, vol. 1, Basel, 1950.

CONCERNING THE FOURIER COEFFICIENTS OF A NONNEGATIVE FUNCTION

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In vol. 66, pp. 223–224 of this Monthly, J. D. Weston established the inequality

$$c_0^2 + c_0|c_{2n}| \ge 2|c_n|^2 \tag{1}$$

for the Fourier coefficients of a nonnegative function $g(\theta)$:

$$g(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in\theta}d\theta. \tag{2}$$

The proof does not seem to point out the basic and simple background of this inequality.

First, let us apply the operation of "symmetrization":

$$g_n(\theta) = \frac{1}{n}\sum_{\nu=0}^{n-1} g\Big(\theta + \frac{2\pi\nu}{n}\Big).$$

It produces a function $g_n(\theta)$ whose harmonics are precisely those of $g(\theta)$ which have a frequency divisible by n:

$$g_n(\theta) \sim \sum_{m=-\infty}^{\infty} c_{nm} e^{inm\theta}.$$

Thus it suffices to prove (1) in the special case $n=1$:

$$c_0^2 + c_0 |c_2| \geqq 2|c_1|^2. \tag{3}$$

We consider the Toeplitz form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,|u_0 + u_1 e^{i\theta} + u_2 e^{i\theta}|^2 d\theta \tag{4}$$

with the matrix

$$\begin{pmatrix} c_0 & c_1 & c_2 \\ c_{-1} & c_0 & c_1 \\ c_{-2} & c_{-1} & c_0 \end{pmatrix}, \qquad c_{-m} = \bar{c}_m. \tag{5}$$

Since (4) is positive definite (unless $g(\theta)$ is a zero-function), we have

$$c_0 > 0, \quad \begin{vmatrix} c_0 & c_1 \\ c_{-1} & c_0 \end{vmatrix} > 0, \quad \begin{vmatrix} c_0 & c_1 & c_2 \\ c_{-1} & c_0 & c_1 \\ c_{-2} & c_{-1} & c_0 \end{vmatrix} > 0. \tag{6}$$

We have the identity

$$c_0 \begin{vmatrix} c_0 & c_1 & c_2 \\ c_{-1} & c_0 & c_1 \\ c_{-2} & c_{-1} & c_0 \end{vmatrix} = \begin{vmatrix} c_0 & c_1 \\ c_{-1} & c_0 \end{vmatrix} \cdot \begin{vmatrix} c_0 & c_1 \\ c_{-1} & c_0 \end{vmatrix} - \begin{vmatrix} c_1 & c_2 \\ c_0 & c_1 \end{vmatrix} \cdot \begin{vmatrix} c_{-1} & c_0 \\ c_{-2} & c_{-1} \end{vmatrix},$$

so that (6) can be written as follows:

$$c_0^2 - |c_1|^2 > |c_1^2 - c_0 c_2|, \qquad c_0 > 0. \tag{7}$$

But $|c_1^2 - c_0 c_2| \geqq |c_1^2| - |c_0 c_2| = |c_1|^2 - |c_0 c_2|$, and this yields (3).

We note that (6) represents the necessary and sufficient condition for the initial Fourier-Stieltjes coefficients of a nonnegative measure $d\alpha(\theta)$:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} d\alpha(\theta). \tag{8}$$

THE INCIDENCE EQUATION $AA^T=aA$

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We say that a matrix is an incidence matrix if each of its elements is either 0 or 1. We denote the zero-matrix of order n by O_n and the one-matrix (the matrix with 1 in every position) of order n by J_n. We shall prove the

THEOREM. *An incidence matrix A satisfies $AA^T=aA$ for some nonzero a if and only if there exists a permutation matrix P such that PAP^T is the direct sum of a zero-matrix and one-matrices of order a.*

The equation $AA^T=aA$ implies that A is square (denote its order by n), and symmetric.

Let $A=(a_{ij})$ and $a_i=\sum_{k=1}^{n} a_{ik}$. The defining equation is equivalent to

$$\sum_{k=1}^{n} a_{ik}a_{jk} = aa_{ij} \qquad i,j = 1, 2, \cdots, n. \tag{1}$$

If we let $i=j$ and note that $a_{ik}^2=a_{ik}$, we have $a_i=aa_{ii}$. Therefore, if the ith row is not a zero row (*i.e.*, if $a_i>0$) then $a_{ii}=1$ and $a_i=a$ or, in words, the nonzero rows of A have row sum a and 1 on the main diagonal.

If $A=0$ we are finished. Suppose $A\neq 0$. We can simultaneously renumber the rows and columns of A (*i.e.*, we can find a permutation matrix Q so that QAQ^T effects the renumbering) in such a way that the first a elements of the first row are 1 and the rest 0. It is important to note that such a renumbering permutes the diagonal elements, so that we can achieve this only because the diagonal element of a non-zero row is 1. Such a renumbering also leaves invariant the defining equation $AA^T=aA$.

Therefore, suppose $a_{11}=\cdots=a_{1a}=1$ and $a_{1k}=0$ for $k>a$. Let $i=1$ and let j lie between 1 and a in equation (1). Then $a_{ij}=1$ and we have

$$\sum_{k=1}^{a} a_{kj} = a. \qquad j = 1, \cdots, a.$$

Since each a_{kj} is either 0 or 1 it follows that $a_{kj}=1$ for $k, j=1, \cdots, a$. Because the row, and by symmetry the column, sums of the nonzero rows (and columns) are equal to a we have $a_{kj}=0$ if $1\leqq k\leqq a$ and $j>a$ or if $1\leqq j\leqq a$ and $k>a$.

Therefore A is the direct sum of J_a and a matrix of order $n-a$ with the same defining equation. We now repeat the above process on this matrix to prove the necessity half of the theorem.

The sufficiency is clear because $O_mO_m^T=aO_m$ for any m, and $J_aJ_a^T=aJ_a$. This completes the proof of the theorem.

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THE PROBABILITY THAT $(n, f(n))$ IS r-FREE

P. J. McCarthy, Florida State University

Let f be a function mapping the set of positive integers into the set of non-negative integers. J. Lambek and L. Moser [1] have shown that with certain conditions* placed on f, the probability that n and $f(n)$ are relatively prime is $6/\pi^2$. This result complements the fact that if the integers m and n are chosen at random then the probability that they are relatively prime is $6/\pi^2$. We have shown ([2], Th. 4) that if m and n are chosen at random then the probability that (m, n) is r-free, *i.e.*, is not divisible by the rth power of any prime, is $1/\zeta(2r)$. In this note we shall show that an analogue to the Lambek-Moser result holds when $r>1$.

THEOREM. *Let $r>1$. Assume that*

(i) *f is nondecreasing,*

(ii) *f^* is finite and nondecreasing, where $f^*(n)$ is the number of m such that $f(m)=n$,*

(iii) $f(n)=o(n)$,

(iv) $f^*(f(n))=o(n)$.

Then, if the integer n is chosen at random, the probability that $(n, f(n))$ is r-free is $1/\zeta(2r)$.

Let $Q(t)$ be the number of positive integers $n \leqq t$ such that $(n, f(n))$ is r-free. We shall show that if f satisfies (i)–(iv) then

$$\lim_{t\to\infty} t^{-1}Q(t) = 1/\zeta(2r).$$

Let $S(t, a, b)$ be the number of multiples of a, not exceeding t, which are mapped onto multiples of b by f. Let μ be the Möbius function. As we have pointed out in [2],

$$\sum_{d^r|n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } r\text{-free,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$Q(t) = \sum_{\substack{n\leq t \\ (n,f(n)),r-\text{free}}} 1 = \sum_{n\leq t} \sum_{d^r|(n,f(n))} \mu(d) = \sum_{d=1}^{k} \mu(d)S(t, d^r, d^r),$$

where we have denoted $[f(t)^{1/r}]$ by k. Lambek and Moser have shown in [1] that as a consequence of (i) and (ii),

$$S(t, a^r, b^r) = (ab)^{-r}t + O(b^{-r}f(t)) + O(a^{-r}f^*(f(t))).$$

Hence,

* Conditions considerably weaker than those of Lambek and Moser have been obtained by P. Erdös and G. G. Lorentz in a paper to appear in Acta Arithmetica.

$$Q(t) = t\sum_{d=1}^{k}\mu(d)d^{-2r} + O\left(f(t)\sum_{d=1}^{k}d^{-r}\right) + O\left(f^*(f(t))\sum_{d=1}^{k}d^{-r}\right).$$

Now,

$$\sum_{d=1}^{k}\mu(d)d^{-2r} = 1/\zeta(2r) + O\left(\sum_{d=k+1}^{\infty}d^{-2r}\right),$$

and it is a simple exercise to show that

$$\sum_{d=k+1}^{\infty}d^{-2r} = O(k^{-r}) \quad \text{and} \quad \sum_{d=1}^{k}d^{-r} = O(1).$$

Therefore,

$$Q(t) = t/\zeta(2r) + O(tf(t)^{-1}) + O(f(t)) + O(f^*(f(t))).$$

It is a consequence of (i) and (ii) that $nf(n)^{-1} = o(n)$, and therefore, using (iii) and (iv) we have $\lim t^{-1}Q(t) = 1/\zeta(2r)$ as $t \to \infty$.

Example. Let $f(n) = [n^{1/s}]$, where s is an integer greater than 1. Then

$$f^*(n) = \sum_{i=1}^{s}\binom{s}{i}n^{s-i},$$

and so f satisfies all the conditions of the theorem.

Examples can be given, similar to those in section 5 of [1], which show that none of the conditions of the theorem are superfluous.

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1. Joachim Lambek and Leo Moser, On integers n relatively prime to $f(n)$, Canad. J. Math., vol. 7, 1955, pp. 155–158.

2. Paul J. McCarthy, On a certain family of arithmetic functions, this MONTHLY, vol. 65, 1958, pp. 586–590.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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ON SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

D. A. KEARNS, Merrimack College

The extent to which the exponential function enters into the study of linear differential equations is familiar to all students of the subject. We point out here that the functional equation satisfied by e^x also can be applied in a certain way

to solutions of nth order differential equations and that in a sense the sine, cosine, and the exponential functions are particular aspects of the same, more general phenomenon.

Consider the differential equation with constant coefficients,

$$\sum_{j=0}^{n} a_j y^{(n-j)}(x) = 0, \tag{1}$$

and the n linearly independent solutions, $y_i(x)$, $i=1, \cdots, n$, determined by the initial conditions,

$$y_i^{(k)}(0) = \begin{cases} 1 & \text{if } k = i-1, \\ 0 & \text{if } k \neq i-1, \end{cases} \tag{2}$$

$i=1, \cdots, n$, $k=0, \cdots, n-1$. Here $y_i^{(0)}(x)=y_i(x)$. Since the a_j are constants, the functions $y_i(x+a)$ are also solutions of (1) for arbitrary values of a. Therefore,

$$y_i(x+a) = \sum_{j=1}^{n} c_{ij} y_j(x), \qquad i = 1, \cdots, n. \tag{3}$$

Differentiating (3) $n-1$ times and making use of (2), it is readily seen that

$$c_{ij} = y_i^{(j-1)}(a), \qquad i, j = 1, \cdots, n. \tag{4}$$

If we now let $y(x)$ represent the column vector whose components are the $y_i(x)$, $i=1, \cdots, n$, then (3) and (4) imply

$$y(x+a) = \phi(a)y(x), \tag{5}$$

where $\phi(x)$ is the matrix with $y_i^{(j-1)}(x)$ in the ith row and the jth column. By successively differentiating (5), it follows that $\phi(x)$ satisfies the identity

$$\phi(x+a) = \phi(a)\phi(x). \tag{6}$$

If (1) has the special form, $y^{(n)}+(-1)^n y=0$, then for $n=1$, (5) and (6) are statements of the fact that $e^{x+a}=e^x e^a$. For $n=2$, $y_1(x)=\cos x$ and $y_2(x)=\sin x$: hence (5) becomes

$$\begin{pmatrix} \cos(x+a) \\ \sin(x+a) \end{pmatrix} = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}.$$

This equation yields the basic trigonometric identities as does, superfluously, (6).

Analogies such as (5) and (6) between solutions of linear equations and the exponential function are suggested by the fact that (1) may be studied by means of systems of first order equations, $y'=Ay$, in which the matrix exponential, e^{Ax}, plays a major role (see, for example, R. Bellman, *Stability Theory of Differential Equations*, New York, 1953).

A NOTE ON THE BASE OF NATURAL LOGARITHMS

ROBERT WEINSTOCK, University of Notre Dame*

Ernest Leach† has exhibited an elegant device for proving that $(1+t)^{1/t} \to e$ as $t \to 0$ *directly* from the definition of e through the equation

$$\int_1^e t^{-1}dt = 1. \tag{1}$$

It is the purpose of the present note to show how Leach's proof can be modified so as to avoid his use of the uniform-continuity concept. (As an inessential alteration we consider here the sequential limit and prove the equivalent $\lim_{n\to\infty}(1+n^{-1})^n = e$.)

We require the following elementary results from the integral calculus of *continuous* functions:

(i) If $f(t) < g(t)$ for $a < t \leqq b$, then $\int_a^b f(t)dt < \int_a^b g(t)dt$.

(ii) If $a < b$, $a < c$, and $0 < f(t) < g(t)$ for $t > a$, then $\int_a^b f(t)dt = \int_a^c g(t)dt$ implies $a < c < b$.

(iii) If $a < b$, $a < c$, and $0 < f(t)$ for $t > a$, then $\int_a^b f(t)dt > \int_a^c f(t)dt$ if and only if $b > c$.

We require also the fact (iv) that if $p > 1$, then for each $u > 0$ there is an $N(u)$ such that $p^{-1/n} > 1 - u$ whenever $n > N(u)$.

Defining (after the manner of [1]) the sequence $\{s_n\}$ by the equation

$$\int_1^{s_n} t^{-1+1/n}dt = 1, \tag{2}$$

we have from (1) and (ii) that $1 < s_n < e$ (and incidentally that $s_{n+1} > s_n$) for all natural numbers n. From (1) and (iii) we have, for arbitrary ϵ satisfying $0 < \epsilon < e - 1$,

$$\int_1^{e-\epsilon} t^{-1}dt = 1 - u(\epsilon), \quad \text{with} \quad u(\epsilon) > 0. \tag{3}$$

From (2) and (i), since $1 < s_n < e$—whence $t^{-1+1/n} < t^{-1}e^{1/n}$ for $1 \leqq t \leqq s_n$—we obtain with the subsequent aid of (iv)

$$1 < e^{1/n}\int_1^{s_n} t^{-1}dt, \text{ whence } \int_1^{s_n} t^{-1}dt > e^{-1/n} > 1 - u(\epsilon) \text{ for } n > N(u(\epsilon)) = N_0(\epsilon).$$

We then conclude from (3) and (iii) that $s_n > e - \epsilon$ for $n > N_0(\epsilon)$; with $s_n < e$, we therefore have $\lim_{n\to\infty} s_n = e$. Since direct computation in (2) yields $s_n = (1+n^{-1})^n$, we have achieved the desired result.

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* On leave at Oberlin College, 1959–60.

† Ernest Leach, A note on the base of natural logarithms, this MONTHLY, vol. 60, 1953, pp. 622–623.

AN EXTENSION OF INTEGRATION BY PARTS

JAMES W. BROWN, Stanford University

The undergraduate mathematics student spends much of his time learning basic formulas which become familiar tools in applications. An example is the method of Integration by Parts which he must often apply a number of times in the same problem. There is a useful extension of this method which can be simply stated and easily remembered.

The basic integration formula is given in the operational notation as

$$D^{-1}(fg') = fg - D^{-1}(gf').$$

If $g = D^{-1}F$, an indefinite integral of the function fF is given by

$$D^{-1}(fF) = fD^{-1}(F) - D^{-1}[D(f)D^{-1}(F)]. \tag{1}$$

It follows by mathematical induction that

$$D^{-1}(fF) = \sum_{i=0}^{n} (-1)^i D^i(f) D^{-(i+1)}(F) - (-1)^n D^{-1}[D^{n+1}(f) D^{-(n+1)}(F)]. \tag{2}$$

Equation (1) is the case for $n = 0$, equation (2) is the inductive hypothesis, and the conclusion is that

$$\begin{aligned} D^{-1}(fF) &= \sum_{i=0}^{n} (-1)^i D^i(f) D^{-(i+1)}(F) \\ &\quad - (-1)^n \{ D^{n+1}(f) D^{-(n+2)}(F) - D^{-1}[D^{n+2}(f) D^{-(n+2)}(F)] \} \\ &= \sum_{i=0}^{n+1} (-1)^i D^i(f) D^{-(i+1)}(F) \\ &\quad - (-1)^{n+1} D^{-1}[D^{n+2}(f) D^{-(n+2)}(F)]. \end{aligned}$$

If f is a polynomial of degree n, $D^{n+1}(f) = 0$ and (2) reduces to the integration formula:

$$D^{-1}(fF) = \sum_{i=0}^{n} (-1)^i D^i(f) D^{-(i+1)}(F). \tag{3}$$

This result may be thought of as an algorithm where the polynomial f is successively differentiated, the function F is successively integrated, alternating signs are affixed to the terms, and an arbitrary constant is added for generality. For example,

$$\begin{aligned} \int (x^2 + x + 1)e^{2x}dx &= D^{-1}[(x^2 + x + 1)e^{2x}] \\ &= (x^2 + x + 1)(\tfrac{1}{2}e^{2x}) - (2x + 1)(\tfrac{1}{4}e^{2x}) + 2(\tfrac{1}{8}e^{2x}) + C. \end{aligned}$$

ERRORS RESULTING FROM A MANIPULATIVE APPROACH TO PROVING THE ORTHOGONALITY OF TWO CURVES

KENNETH MAY, Carleton College

If asked to prove that the two curves defined by $y = -\sqrt{(2x)}$ and $y = e^{-x}$ are orthogonal, the thoughtless manipulator quickly finds $y' = 1/y$ for the first and $y' = -y$ for the second. Citing something about "negative reciprocals" he announces that the proof is complete. Protesting that it is really a waste of time, he draws the curves—and discovers that they do not meet at all! The experience may help convince him that formal calculations are not enough. It is easy to find other examples by experiment. A routine procedure (not available to the student in beginning calculus) is to construct the family of orthogonal trajectories of a family of curves and then pick a nonintersecting pair from the two families.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

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AAAS STUDIES IN TEACHER EDUCATION

The educational programs of the National Science Foundation and of scientific societies in cooperation with NSF, with private foundations, or with their own resources, indicate widespread interest and progress in the improvement in offerings in science and mathematics for those preparing to teach these subjects in elementary and secondary schools. On the other hand, not much information is available about studies of the professional education requirements for science teachers. As a consequence, one of the important projects in the new Science Teaching Improvement Program (STIP) of the American Association for the Advancement of Science is entitled "Studies in Teacher Education." The project is intended to encourage experimentation with professional requirements for science teachers. An advisory board has chosen seven colleges and universities for cooperation in the project. These colleges and universities, with a brief description of the projects they are undertaking, are listed below.

For the most part, the studies will be carried out independently by the seven colleges, but there may be advantages in the cooperation of two or more of the institutions of higher education on a problem of common interest. A part of the STIP contribution will be to make this possible. A first conference of representatives of the cooperating institutions was held at the University of Michigan on July 20–21. A feature of that conference was a report from the West Virginia

State Department of Education on the plans in that state for teacher certification by examination. A brochure announcing the new studies, containing information about them and the cooperating institutions is available for distribution on request.

The cooperating colleges and their projects are as follows:

Bucknell University and Emory University—Use of examinations to satisfy teacher-certification requirements.

Emory University and the University of Tennessee—Use of seniors in college laboratories as part of the student-teaching assignment and as assistants in research projects in science.

Hunter College and Emory University—Development of a course in science (Hunter) and a course in mathematics (Emory) for elementary-school teachers, to be offered jointly by science and education staffs.

Oklahoma State University—An experimental study in the education of elementary teachers, under which elementary teachers will do their student teaching during the second semester of the junior year with the advice and counsel of science staff members. During the senior year science staff members will further assist them in making appropriate choices of content courses to supplement the four-year bachelor's degree program.

San Francisco State College—The establishment of a laboratory of materials for teaching science and mathematics, and participation, as a part of the professional education requirements for prospective teachers, in keeping the laboratory up to date and making use of the materials.

University of Arizona—A part of requirement for preservice science (including mathematics) teachers will be a seminar of scientists and preservice teachers, at which papers will be presented and discussed both from the scientific and pedagogical points of view.

GEOMETRY FOR ELEMENTARY SCHOOL TEACHERS

LENORE JOHN, The Laboratory School, University of Chicago

Among the proposals for revision of the arithmetic curriculum in the elementary grades is the suggestion that study of some aspects of mathematics other than arithmetic be begun in those grades. In particular, it is proposed that some geometry be included. Studies of the development of young children indicate that some space concepts are developed early. The everyday experiences of children contribute to further development, and suggest the possibility of a more systematic treatment of space relations than is currently found in the curriculum. Below the seventh grade, the geometry content is frequently limited to study of the perimeter and the area of a rectangular region and the volume of a rectangular solid, with perhaps some study of the triangle. Exploratory studies now in progress seem to indicate that other topics can be taught with success even in the primary grades, and it seems probable that new courses of study will include more geometry than has been the case in recent years.

Exactly what aspects of geometry should become part of the curriculum for Grades One to Six is a question which must be answered in the light of evidence obtained from experimental teaching in classrooms, as well as other considerations. However, successful teaching of geometric content on any widespread scale will depend on adequate preparation of teachers. Courses designed to provide such preparation may well include at least the following topics and points of emphasis.

1. Consideration of points, lines, planes, and space as sets of points, and their intersections and unions. Conditions determining a point, a line, a plane. Separation of a line, a plane, and space. Simple closed curves and their intersections; non-metric properties of common polygons and polyhedrons. Concepts associated with projection in a plane and in three dimensions.

2. Correspondence between the set of points on a line and the set of real numbers, between the points in a plane and ordered pairs, between points in space and ordered triples. Geometric and algebraic description of geometric figures, including the line, circle, ellipse, and parabola.

3. Assumptions underlying the measurement of length, area, volume, and angle. The concept of precision of a measure.

4. Angle relationships associated with intersecting and parallel lines and planes. Congruence and similarity of geometric figures. Metric properties of common two- and three-dimensional figures, including the triangle, quadrilateral, circle, prism, pyramid, cylinder, cone, and sphere. The special properties of the right triangle.

5. Physical representations of geometric figures in the environment; description of familiar objects in terms of their geometric properties; application of geometric relations in solving practical problems.

6. Presentation of the geometric content in such a way as to make clear the use of inductive reasoning in drawing a probable inference from observed data; the use of deductive reasoning in drawing a necessary inference from stated postulates; and the need for undefined terms and for precisely stated postulates and definitions.

7. Brief overview of the history of the development of geometry.

The amount of the content outlined above which is new to the prospective teacher will depend in part upon the amount of mathematics studied in the secondary school. One who has studied two or three years of college preparatory mathematics should be familiar with much of the material, but even in such cases a review of the topics from a new point of view would probably be desirable.

While adequate mathematical background is the most important element in preparation of teachers to introduce geometry in the elementary grades, assistance is also needed in developing effective methods of teaching, especially in the selection and construction of suitable models. If a mathematics course and a course in methods of teaching are taught by different instructors, close collaboration between instructors is essential for effective preparation of the prospective teachers.

NSF Program in Mathematics for High-Ability Secondary School Students at the University of Texas

During the summer of 1959 the University of Texas sponsored a program in mathematics for high-ability secondary school students. Thirty participants were selected from a total of 548 applications. The six-week program consisted of daily group meetings and work sessions.

In the report of the Director, H. J. Ettlinger, Professor of Mathematics, the contents were described as follows:

> "The objective of the program was to provide a maximum opportunity for the participants to broaden and deepen their knowledge of mathematics. In the main the assumption was made that this could be accomplished best not by reading or listening to lectures, but by *working problems* and *doing mathematics*. In the main a definite attempt was made to avoid those mathematical subjects which the pupils would be studying the following school year. As evidenced by the summary and notes in the appendix, attention was concentrated on the basic idea of number in arithmetic and algebra, and developing fundamental mathematical patterns by working with theorems and exercises. This was also followed in geometry, which was developed by means of sets of points, definitions and axioms, the very essence of modern mathematics."

And he further reported that:

> "The pupils responded to this program far more enthusiastically and successfully than could have been expected. It was very heartening to witness the presentation of mathematical ideas and proofs of this high-ability group. This they did with equal success in algebra and in geometry. They handled the patterns of modern mathematics with great skill. . . . This program has been a most enjoyable and exciting experience for the staff. It is hoped that other opportunities to work with groups of pupils of this type will be provided."

GREATER CLEVELAND MATHEMATICS PLAN

GEORGE H. BAIRD, The Educational Research Council of Greater Cleveland

In September 1959, 20 public, private and parochial schools from the Greater Cleveland area (Cleveland City Public Schools not included) started implementing a new mathematics curriculum called the Greater Cleveland Mathematics Program (G.C.M.P.). This mathematics curriculum project was initiated by the Educational Research Council of Greater Cleveland and is directed by Dr. B. H. Gundlach of Bowling Green State University. The G.C.M.P. provides a five year program with new curricula introduced gradually in all grades of elementary and secondary schools after transitional and enrichment programs and in-service teacher training.*

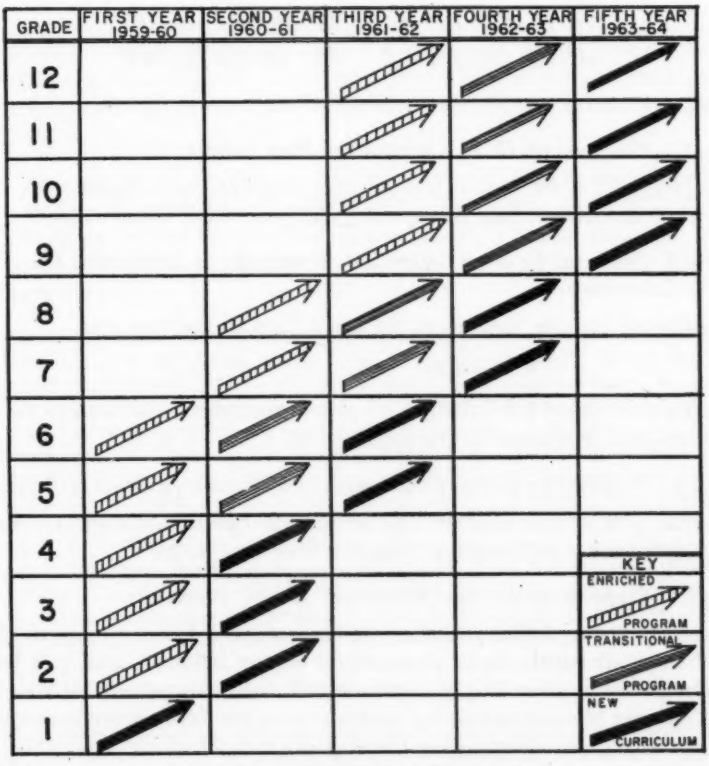
For the first school year, 1959–60, during the first semester, an intensive 1st-grade-teacher retraining is provided by Dr. Gundlach and the Council staff. The new 1st-grade-curriculum is being introduced to the pupils in the second semester. Meanwhile teachers from kindergarten through grade 6 were given in-

* Added in proof: The completed G.C.M.P. will result in one continuous mathematics course, starting in kindergarten and going through grade 12, supplemented by an advanced placement program. The schedule of implementation shown in the diagram is now somewhat out of date, since it does not include the kindergarten grade and certain other modifications.

service training so that they could introduce in their classrooms an enriched course of study. In the following four years the new curriculum will reach all 13 grades, kindergarten through grade 6.

The Council has received the collaboration of Dr. J. G. Kemeny, Dartmouth College; Dr. Max Beberman, University of Illinois; Dr. C. F. Brumfiel, Ball State Teachers College; and Dr. P. S. Jones, University of Michigan, to insure a first hand combination of the best of all recent research in the field of mathematics.

The prompt implementation of the G.C.M.P. by a compact group of 20 schools, with a school enrollment of 100,000 pupils, will offer an opportunity for close supervision and evaluation of the whole program, so that if successful in this limited area, it could serve as a pattern for other schools in the nation to follow.



PROPOSED SCHEDULE OF IMPLEMENTATION

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1411. *Proposed by H. Y. Shee, Hualien, Taiwan, China*

Let $ABCDE$ be any pentagon inscribed in a circle and let P, Q, R, S, T be intersections of the diagonals such that P and Q lie on AC, Q and R on BD, R and S on CE, S and T on DA. Prove that

$$\frac{AB\cdot BC\cdot CD\cdot DE\cdot EA}{AC\cdot BD\cdot CE\cdot DA\cdot EB}=\frac{AP\cdot BQ\cdot CR\cdot DS\cdot ET}{CP\cdot DQ\cdot ER\cdot AS\cdot BT}.$$

Generalize.

E 1412. *Proposed by H. E. Chrestenson, Reed College*

Let $\{m_k\}$, $k\geqq 0$, be a strictly increasing sequence of integers with $m_0=0$. Under what conditions does the series $\sum_{k=1}^{\infty}(m_k-m_{k-1})/m_k$ converge?

E 1413. *Proposed by B. K. Harrison, University of California, Los Alamos Scientific Laboratory*

Consider an integer, written in the usual form to the base r as

$$N = a_0a_1a_2\cdots a_n, \qquad 0\leqq a_i<r.$$

We require that $r\geqq n+1$. Find all N, for any n, such that a_k is equal to the number of times that k appears in the sequence $a_0, a_1, a_2, \cdots, a_n$.

E 1414. *Proposed by George Glauberman, Polytechnic Institute of Brooklyn*

Suppose g is a nonnegative and integrable function on $[0, 1]$. Let $A_r=\int_0^1[g(x)]^r dx$ for r a nonnegative integer. Prove that $A_r\geqq A_1^r$.

E 1415. *Proposed by Michael Herschorn, McGill University*

It began snowing in the morning; at noon a snowblower set out to clear a street of constant width. Later on a second blower followed, and, still later, a third. Each blower cleared at a constant rate (volume per unit of time), the spacing between blowers remaining constant with the first separation twice the second. At any given time during the day the rate of snowfall was the same throughout the town. A total of four inches had fallen when the second blower started out, and another four inches fell before the third began on the route. How deep was the snow at noon?

SOLUTIONS

A Polynomial Equation with no Real Roots

E 1381 [1959, 724]. *Proposed by H. E. Hart, City College of New York*

Show that

$$1 + x + x^2/2! + x^3/3! + \cdots + x^{2n}/(2n)! = 0$$

has no real roots.

I. *Solution by Joe Lipman, University of Toronto.* If there is a real root, it must be negative, say $-y$. But

$$1 - y + y^2/2! - y^3/3! + \cdots + y^{2n}/(2n)! > e^{-y} > 0.$$

II. *Solution by H. F. Bechtell, Grove City College.* Note that if $f(x) = f'(x) + x^{2n}/(2n)! = 0$, then $f'(x) < 0$; *i.e.*, at each root the function is decreasing. Since $f(x)$ is continuous and positive for all $x \geqq 0$, a contradiction is reached. Thus $f(x)$ has no real roots.

III. *Solution by C. F. Pinzka, University of Cincinnati.* Denoting the left member of the equation by $f(x)$, we note that $f(x) = f'(x) + x^{2n}/(2n)!$. Since $f(x)$ is of even degree with a positive leading coefficient, it has an absolute minimum at, say, $x = a$, where $a \neq 0$ and $f'(a) = 0$. Thus $f(a) = a^{2n}/(2n)! > 0$ and $f(x)$ is never zero for real x.

Also solved by R. G. Albert, Tom Bagby, Edward Barbeau, Merrill Barnebey, P. T. Bateman, W. R. Becker, Shlomo Ben-Adam, Marvin Blum, D. A. Breault, Brother Joseph Heisler, J. L. Brown, Jr., P. R. Chernoff, C. H. Cunkle, A. E. Danese, Underwood Dudley, G. W. Erwin, Jr., N. J. Fine, Evelyn Frank, Stanley Franklin, W. C. G. Fraser, J. D. Gilbert, T. A. Gitlin, George Glauberman, Gerald Goertzel, Michael Goldberg, L. D. Goldstone, S. H. Greene, Emil Grosswald, D. W. Henderson, J. C. Hickman, J. H. Hodges, Richard Holt, I. M. Isaacs, Lawrence Isenecker, J. N. Issos, Walter James, G. J. Janusz, C. M. Joiner, J. P. Jordan, Seymour Kass, Irving Katz, P. G. Kirmser, D. E. Knuth, Andrew Korsak, A. M. Krall, W. E. Lawrence, W. J. Larkin III and L. J. Schneider (jointly), Joel Levy and H. J. Lieberman (jointly), R. J. Lewyckyj, Viktors Linis, Shen Lui, R. F. McDermot, W. M. McKeeman, Wallace Manheimer, R. H. Marquis, D. C. B. Marsh, D. S. Mitrinović, Morris Morduchow, T. J. Morse, Amos Nannini, Hugh Noland, C. S. Ogilvy, Henry Palczewski, D. J. Persico, J. L. Pietenpol, Thomas Porsching, P. L. Renz, R. T. Sandberg, Donna Jean Seaman, Harold Shulman, E. L. Spitznagel, Jr., D. C. Stevens, J. P. Tull, K. Venkannayah, J. E. Vinson, Julius Vogel, R. J. Wagner, Dale Woods, Kenneth Williams, C. Winston, R. J. Wisner, J. D. M. Wright, C. C. Yalavigi, Catherine Yamada, David Zeitlin, and the proposer. Late solutions by M. S. Klamkin, Nathaniel Queen, B. E. Rhoades, and Jonathan Sondow.

This problem was located, with solution, in Polyá-Szego, *Aufgaben und Lehrsätze aus der Analysis*, Zweiter Band, Prob. 74, p. 48, with a further reference to J. J. Sylvester, *Mathematical Papers*, vol. 2, p. 516. It also appears, with solution, in E. A. Maxwell, *An Analytical Calculus*, vol. 1, p. 65, and in D. S. Mitrinović, *Zbornik matematičkih problema*, t. 1, 3° ed. (in press), and as part of Prob. 9, p. 118, of J. V. Uspensky, *Theory of Equations*.

An Unusual Sequence

E 1382 [1959, 724]. *Proposed by Ian Connell, University of Manitoba*

Let u_n be the nth term of the sequence

$$1, 2, 4, 5, 7, 9, 10, 12, 14, 16, 17, \cdots,$$

where one odd number is followed by two evens, then three odds, etc. Prove that

$$u_n = 2n - [(1 + \sqrt{8n - 7})/2],$$

where square brackets denote the integral part.

Solution by Andrew Korsak, University of Toronto. Writing $u_n = 2n - t_n$, we see that the sequence $\{t_n\}$ is

$$1; 2, 2; 3, 3, 3; 4, 4, 4, 4; 5, 5, \cdots.$$

For a given t_n, it follows that the value of $n-1$ cannot be less than the sum of the first $t_n - 1$ integers, hence $t_n(t_n-1)/2 \leqq n-1$, from which we obtain $t_n \leqq (1+\sqrt{8n-7})/2$ and, t_n being the largest integer to satisfy this, we conclude $t_n = [(1+\sqrt{8n-7})/2]$.

Also solved by R. G. Albert, Edward Barbeau, Shlomo Ben-Adam, Julian Braun, J. L. Brown, Jr., Leonard Carlitz, G. B. Charlesworth A. G. Clark, Underwood Dudley, N. J. Fine, George Glauberman, H. W. Gould, Alfred Gray, S. H. Greene, Emil Grosswald, J. H. Hodges, J. Hooley, John Jordan, P. G. Kirmser, D. E. Knuth, Sidney Kravitz, H. R. Leifer, R. J. Lewyckyj, Joe Lipman, Shen Liu, William McKeeman, Wallace Manheimer, D. C. B. Marsh, N. S. Mendelsohn, Amos Nannini, Walter Penney, C. F. Pinzka, C. M. Sandwick, Sr., J. K. Siberz, C. M. Sidlo, Philip Smedley, D. C. Stevens, J. P. Tull, K. Venkannayah, J. E. Vinson, Chung-lie Wang, C. Winston, J. D. M. Wright, C. C. Yulavigi, and the proposer. Late solutions by M. S. Klamkin and Nathaniel Queen.

Mendelsohn pointed out that if v_n is the nth term of the sequence "complementary" to $\{u_n\}$, i.e., the sequence of integers in increasing order of magnitude which do not appear in $\{u_n\}$, namely, 3, 6, 8, 11, 13, 15, 18, 20, 22, 24, $\cdots$, then, like $\{u_n\}$, $\{v_n\}$ consists of one odd, two even, three odd numbers, etc. The only difference in rule is that in changing parity the first available integer is omitted. Also, $u_n + v_n = 4n$.

Gould noted the connection of the problem with that of the countability of the rational numbers, and he explored some generalizations of the problem.

A Problem with a Unique Solution

E 1383 [1959, 724]. *Proposed by E. J. Burr, University of New England, N.S.W., Australia*

Let P_n be the problem: "Find a set of $n>2$ consecutive positive integers, the greatest of which is a divisor of the least common multiple of the remaining $n-1$." Show that precisely one of the problems P_n has a unique solution.

Solution by N. J. Fine, Institute for Advanced Study. For fixed n, let M_b be the least common multiple of $b-1, b-2, \cdots, b-(n-1)$. Since every prime power which divides b and some j, $1 \leqq j < n$, also divides b and $b-j$, and con-

versely, $(b, M_b) = (b, M_n)$. In particular, $b \mid M_b$ is equivalent to $b \mid M_n$. Thus, the solutions of P_n are precisely those b which divide M_n and are not less than n. For $n=3$, $M_n = 2 < n$, so P_3 has no solution. For $n=4$, $M_4 = 6$, and P_4 has a unique solution. For $n \geqq 5$, both $(n-1)(n-2)$ and $(n-1)(n-2)/2$ are divisors of M_n which are not less than n, hence solutions of P_n.

Also solved by R. G. Albert, Tom Bagby, Edward Barbeau, G. B. Charlesworth, Underwood Dudley, Merritt Elmore, Michael Goldberg, Donald Knuth, Andrew Korsak, Sidney Kravitz, Emma Lehmer, Joe Lipman, Wallace Manheimer, D. C. B. Marsh, Hugh Noland, J. L. Pietenpol, R. T. Sandberg, B. L. Schwartz, Donna Jean Seaman, D. C. Stevens, J. P. Tull, J. E. Vinson, and the proposer.

Circle Through Two Points Cutting a Given Circle at Least Angle

E 1384 [1959, 724]. *Proposed by J. H. Butchart, Arizona State College*

Construct a circle through two given points, separated by a given circle, which shall cut the given circle at the smallest possible angle.

Solution by D. C. Stevens, Alabama College. Designate the given points by P_1, P_2 and the given circle by C_1. Let C_2 be the circle orthogonal to C_1 and on P_1, P_2, and let C_3 be the circle orthogonal to C_2 and on P_1, P_2. Invert the figure about P_1, designating the images of elements with primes. Now C_2' is a diameter of C_1', and C_3' is the line through P_2' making minimum angle with C_1', since C_3' is perpendicular to C_2'. Angles are preserved under inversion, and so C_3 makes minimum angle with C_1.

Also solved by J. Basile, Michael Goldberg, L. D. Goldstone, Andrew Korsak, R. J. Lewyckyj, D. C. B. Marsh, Beckham Martin, Vincent Mead, C. F. Pinzka, and the proposer. Late solution by J. W. Baldwin.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4899. *Proposed by R. C. Buck, Institute for Advanced Study*

As $(x, y) \to (0, 0)$ let $\lim F(x, y) = 0$. Suppose that $\{s_n\}$ is a real sequence such that $0 \leqq s_{n+m} \leqq F(s_n, s_m)$ for all $n, m = 1, 2, \cdots$. Suppose also that $\lim (s_1 + \cdots + s_n)/n = 0$. Show that $\lim s_n = 0$.

4900. *Proposed by S. P. Lloyd, Bell Telephone Laboratories*

For every measurable subset E of the unit interval let $\{C_n(E)\}$ denote the sequence of numbers

$$C_n(E) = \int_E e^{2\pi i n x} dx, \qquad n = 1, 2, \cdots.$$

Show that a sequence $\{C_n(E)\}$ may tend to zero arbitrarily slowly, in the following sense: for every positive nondecreasing sequence $\{a_n\}$, if $\sup_n a_n |C_n(E)|$ is finite for every E then $\sup_n a_n$ is finite. (Problem 4749 [1958, 453], treats the case $a_n = n$.)

4901. *Proposed by Albert Wilansky, Lehigh University*

Let the class of subsets of measure zero of the interval $[0, 1]$ be partially ordered by inclusion and let M be a maximal chain. Is the cardinal number of M greater than that of the continuum? Is the union of the sets in M measurable?

4902. *Proposed by R. A. Melter, University of Missouri*

At most how many different values may an nth order determinant have, if its elements are a given set of n^2 different, nonzero real numbers?

4903. *Proposed by Victor Klee, University of Washington and University of Copenhagen*

A Darboux map of one topological space onto another is one under which the image of a connected set is always connected. With R denoting the real number space, show that there is a map f of R onto R such that f is a Darboux map but the product g of f with the identity map is not. (Here g is a map of the plane $P = R \times R$ onto itself, given by $g(x, y) = (fx, y)$.)

4904. *Proposed by D. J. Newman, Brown University*

Suppose a set, S, of functions continuous on $(0, 1)$ has the property that any linear combination of them has a zero in $(0, 1)$. Prove that there exists a nondecreasing function $\alpha(x)$ such that

$$\int_0^1 f(x) d\alpha(x) = 0 \qquad \text{for all } f \in S.$$

SOLUTIONS

A Diophantine Equation

4807 [1958, 633]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the integral solutions of the equation

$$(m - 1)x^2 + my^2 = (m + 1)z^2.$$

Editorial Note II. The proposer's solution [1959, 518] is not complete as mentioned in the

earlier *Note*. This is shown by the example: $m=1322$, $x=105$, $y=21$, $z=107$. Nevertheless, since $1322=31^2+19^2$ only, there exist no values of x, y, z such that $x^2+z^2=m$, $x^2+y^2-z^2=1$.

The Ring of p-adic Integers

4820 [1958, 779]. *Proposed by K. Mahler, the University, Manchester, England*

Let p be a prime, K the p-adic field, and I the ring of p-adic integers. Further, let $f(x)$ be a function defined for all $x\in I$ with $f(x)\in K$, such that $f(n)=n!$ $(n=0, 1, 2, \cdots)$. Show that the points of discontinuity lie dense in I.

Solution by Leonard Carlitz, Duke University. Let $P=(0, 1, 2, \cdots)$, the set of nonnegative integers. We show first that $f(x)$ is discontinuous for every $x\in P$. Indeed, for every such x, consider the sequence

$$x_n = x + p^n \qquad (n = 0, 1, 2, \cdots).$$

Then, if $v(x)=v_p(x)$ denotes the usual p-adic valuation for K, we have $v(x_n-x)=v(p^n)\to 0$ $(n\to\infty)$. On the other hand, since $f(x_n)=(x+p^n)!$ it follows that $v(f(x_n))\to 0$ as $n\to\infty$. But since $v(f(x))\neq 0$, we conclude that $f(x)$ is discontinuous at x.

Finally, if $\alpha=a_0+a_1p+a_2p^2+\cdots$ $(0\leqq a_i<p)$ is any number of I, and we put $\alpha_n=a_0+a_1p+\cdots+a_np^n$, then clearly $v(\alpha-\alpha_n)\to 0$ as $n\to\infty$. Hence P is dense in I.

Also solved by P. T. Bateman, Harley Flanders, and the proposer.

Normal Completion of Matrices

4821 [1958, 779]. *Proposed by M. P. Drazin, RIAS, Baltimore, Md.*

A square matrix B is called normal if its elements are in the complex field and if it commutes with its transposed conjugate matrix B^* (*i.e.*, if $BB^*=B^*B$); more generally, call a given $n\times n$ matrix A m-normal if A can be imbedded, as leading $n\times n$ submatrix, in some normal $m\times m$ matrix B. Find the greatest value of n for which every (complex) $n\times n$ matrix is $(n+1)$-normal, and show that every square matrix is m-normal for all sufficiently large m.

What are the corresponding results when A, B are restricted to be real?

Solution by the proposer. We show first that, for each integer $n\geqq 3$, a real $n\times n$ matrix A exists having no normal $(n+1)\times(n+1)$ completion

$$B = \begin{pmatrix} A & x^* \\ y & z \end{pmatrix}$$

(where x, y denote arbitrary complex row n-vectors and z can be any complex number), *i.e.*, an A exists such that

$$(1)\quad BB^* - B^*B = \begin{pmatrix} AA^* - A^*A + x^*x - y^*y, & Ay^* - A^*x^* + x^*\bar{z} - y^*z \\ yA^* - xA + zx - \bar{z}y, & yy^* - xx^* \end{pmatrix}$$

is nonzero for all choices of x, y, z. Indeed, if we choose A to be the $n\times n$ matrix U_n having s as its $(s, s+1)$ elements $(s=1, 2, \cdots, n-1)$ and zeros elsewhere, then in fact $U_nU_n^*-U_n^*U_n+x^*x-y^*y$ cannot vanish. For

$$C \equiv U_nU_n^* = U_n^*U_n = \text{diag. } (1, 3, \cdots, 2n-3, -(n-1)^2),$$

and so, if $n \geqq 3$, then, for arbitrary row vectors $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n)$, the leading 2×2 submatrix of $C+x^*x-y^*y$ is just

$$\begin{pmatrix} 1+|x_1|^2-|y_1|^2, & \bar{x}_1x_2-\bar{y}_1y_2 \\ \bar{x}_2x_1-\bar{y}_2y_1, & 3+|x_2|^2-|y_2|^2 \end{pmatrix}.$$

Hence $C+x^*x-y^*y=0$ would imply

$$(1+|x_1|^2)(3+|x_2|^2) = |y_1|^2|y_2|^2 = |\bar{y}_1y_2|^2 = |\bar{x}_1x_2|^2 = |x_1|^2|x_2|^2,$$

i.e., $3+3|x_1|^2+|x_2|^2=0$, which is of course impossible; in other words, U_n cannot be imbedded in any $(n+1)\times(n+1)$ normal matrix $(n=3, 4, \cdots)$.

We show next that this result is best possible, *i.e.*, (the 1×1 case being entirely trivial) that every 2×2 matrix A can be imbedded in some 3×3 normal matrix B. Indeed, if we take

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} A & x^* \\ y & z \end{pmatrix}, \qquad x=(\bar{c}, u), \qquad y=(b, v), \qquad z=d,$$

where u and v are to be determined, then a direct computation shows that

$$BB^*-B^*B = \begin{bmatrix} 0 & \theta & -\bar{\theta} \\ \bar{\theta} & \phi & 0 \\ -\theta & 0 & -\phi \end{bmatrix},$$

where $\theta=cu-\bar{b}v-f$, $f=\bar{a}b+\bar{c}d-a\bar{c}-b\bar{d}$, $\phi=|u|^2-|v|^2-|b|^2+|c|^2$. Thus we need now only to show that, given any complex a, b, c, d, then complex u, v can be found such that

$$cu-\bar{b}v=f, \qquad |u|^2-|v|^2=|b|^2-|c|^2. \tag{2}$$

If $b=0$, $c\neq 0$, we easily find real solutions v on taking $u=f/c$; and similarly if $b\neq 0$, $c=0$. Also if $f=0$, we can take $u=\bar{b}$, $v=c$. Thus (since $b=c=0$ implies $f=0$) we may suppose $bcf\neq 0$, and write

$$u=\tfrac{1}{2}(\alpha+1)f/c, \qquad v=\tfrac{1}{2}(\alpha-1)f/\bar{b}.$$

For real α, since $bc\neq 0$, these provide a solution of (2) whenever

$$(|b|^2-|c|^2)|f|^2\alpha^2+2(|b|^2+|c|^2)|f|^2\alpha + (|b|^2-|c|^2)(|f|^2-4|b|^2|c|^2)=0;$$

and, since $f\neq 0$, this equation always has at least one real root α.

That every square matrix A is m-normal for large m follows at once from the identity

$$\begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}\begin{pmatrix} A^* & A \\ A & A^* \end{pmatrix} = \begin{pmatrix} A^* & A \\ A & A^* \end{pmatrix}\begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}.$$

Finally, just the same results hold good over the real field; for each U_n is real, and every B we have constructed is real when the corresponding A is.

Hilbert Space, Identity Operator as a Sum

4825 [1959, 66]. *Proposed by J. J. Schäffer, University of Uruguay*

Prove that in any n-dimensional (real or complex) Euclidean space, $n>1$, and indeed in any Hilbert space, every point x, $\|x\| \leqq 1$ may be written $x=\sum_{i=1}^{m} f_i(x)$, where $f_1(x), \cdots, f_m(x)$ is a fixed finite set of functions which are defined and uniformly continuous in $\|x\| \leqq 1$ and satisfy $\|f_i(x)\| = 1$ for $i=1, \cdots, m$ and for all $\|x\| \leqq 1$.

From this result it follows at once that if $\mathfrak{C}$ is the Banach space of all bounded continuous functions $\phi(t)$ of a real variable t with values in any Hilbert space of dimension >1 (with the norm of the supremum), the set of $\phi(t) \in \mathfrak{C}$ with $\|\phi(t)\| = 1$ identically contains a linear basis of $\mathfrak{C}$.

Solution by the proposer. We prove that the theorem indeed holds with $m=4$ in every case. Choose two mutually orthogonal unit vectors e_1, e_2 (this requires dimension >1). For every x, $\|x\| \leqq 1$, set $y(x)=x-(x, e_1)e_1$. Thus $(y(x), e_1)=0$ and $\|y(x)\|^2+|(x, e_1)|^2=\|x\|^2 \leqq 1$. $y(x)$ is a projection, hence uniformly continuous. The functions

$$\begin{aligned} f_1(x) &= \tfrac{1}{2}\{(x, e_1)e_1 + \sqrt{(4 - |(x, e_1)|^2)}\cdot e_2\}, \\ f_2(x) &= \tfrac{1}{2}\{(x, e_1)e_1 - \sqrt{(4 - |(x, e_1)|^2)}\cdot e_2\}, \\ f_3(x) &= \tfrac{1}{2}\{y(x) + \sqrt{(4 - \|y(x)\|^2)}\cdot e_1\}, \\ f_4(x) &= \tfrac{1}{2}\{y(x) - \sqrt{(4 - \|y(x)\|^2)}\cdot e_1\}, \end{aligned}$$

then clearly satisfy the requirements (uniform continuity is implied by the inequalities $4-|(x, e_1)|^2 \geqq 3$, $4-\|y(x)\|^2 \geqq 3$.)

It would be interesting to know if m can ever be chosen <4. For $m=2$, continuity cannot be assured at the origin; for n odd and $m=2$, continuity cannot even be assured if a neighborhood of the origin is excluded.

Also solved by R. D. Gordon.

Convolution, Inner Product

4849 [1959, 427]. *Proposed by L. A. Rubel, University of Illinois*

Suppose that $\phi(t)$ is a bounded measurable real function defined for $t>0$, and let $f(x)=(\phi * \phi)(x)=\int_0^x \phi(t)\phi(x-t)dt$. Must $f(x)$ be nonnegative for all sufficiently small positive x?

Solution by Brockway McMillan, Bell Telephone Laboratories. The answer "no" is indicated by the following example. Define $\phi(t)$ by

$$\phi(t) = (-1)^n \text{ for } 2^{-n-1} < t \leqq 2^{-n}, \qquad n = 0, 1, 2, \cdots.$$

With $f(x)$ as defined in the proposal, it is easy to establish the following three results in the order given:

$$(1) \qquad \phi(\tfrac{1}{2}t) = -\phi(t), \qquad 0 < t \leqq 1;$$

$$(2) \qquad f(\tfrac{1}{2}x) = \tfrac{1}{2}f(x), \qquad 0 \leqq x \leqq 1;$$

$$(3) \qquad f(1) = 2\int_0^{1/2} \phi(t)\,dt = -\tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots = -\tfrac{1}{3}.$$

It follows from (2) and (3) that $f(2^{-n}) < 0$ for $n = 0, 1, \cdots$. Therefore any interval $0 < x \leqq t$ contains points where $f(x) < 0$.

Also solved by N. J. Fine and M. Jerison, Leopold Flatto, E. N. Gilbert, W. M. Gilbert, L. S. Kennison, R. H. Mihalek, John Rainwater, E. D. Ramer, I. J. Schoenberg, B. L. Schwartz, and the proposer.

Editorial Note. Schoenberg obtains similar results using the *continuous* function $\phi(t) = \sin(1/t)$.

Closed Convex Sets

4850 [1959, 427]. *Proposed by Ward Cheney and Allen Goldstein, Convair-Astronautics, San Diego, California*

Let $\mathfrak{A}$ and $\mathfrak{B}$ denote two closed convex sets in Hilbert space. For a point x, let Ax and Bx denote respectively the points of $\mathfrak{A}$ and $\mathfrak{B}$ closest to x. Prove that A satisfies the Lipschitz condition $\|Ax - Ay\| \leqq \|x - y\|$. Furthermore, if x is a fixed point of AB, then $\|x - Bx\| = \text{dist}\,(\mathfrak{A}, \mathfrak{B})$.

Solution by Robert C. James, Harvey Mudd College. Let T be a normed linear space. Then there is a nearest point Ax for each x and closed convex set A if and only if T is reflexive, since this is true for each separable subspace of T. [James, *Reflexivity and the supremum of linear functionals*, Annals of Math., 66 (1957), Theorem 3.] However, these propositions have meaning for T whether Ax necessarily exists and even if it need not be unique. Let $x \perp y$ mean that $\|x + ky\| \geqq \|x\|$ for each k. The first proposition is true if and only if

(i) $x + y \perp z$ whenever $x \perp z$ and $y \perp z$, and

(ii) $y \perp x$ whenever $x \perp y$.

Assume (i) and (ii) and let C be the entire line passing through Ax and Ay. Then two-dimensional considerations of convexity show that $\|Ax - Ay\| \leqq \|Cx - Cy\|$. But $x - Cx \perp Cx - Cy$ and $Cy - y \perp Cx - Cy$. Hence $(x - y) - (Cx - Cy) \perp Cx - Cy$ and $Cx - Cy \perp (x - y) - (Cx - Cy)$. Then

$$\|x - y\| = \|Cx - Cy + [(x - y) - (Cx - Cy)]\| \geqq \|Cx - Cy\| \geqq \|Ax - Ay\|.$$

If T is a Hilbert space, (i) and (ii) are satisfied and the first proposition is established. This was done by Phelps in a different way [*Convex sets and nearest points*, Proc. A.M.S., 8 (1957), Theorem 5.1]. Phelps also showed that (ii) is a

necessary condition [Lemma 5.4], but (ii) is also a sufficient condition for T to be a Hilbert space if the dimension of T is greater than 2. If T is two-dimensional, (i) is equivalent to strict convexity (each is equivalent to the set of all x with $x \perp y$ being one-dimensional when $y \neq \theta$) and the necessity then follows from a result of Phelps [Lemma 5.3].

The second proposition is true if and only if

(iii) $x \perp y+z$ whenever $x \perp y$ and $x \perp z$.

It is known that (iii) is equivalent to the uniqueness of supporting hyperplanes at points of the unit sphere and to the existence of $\lim_{s \to 0} (\|u+sv\| - \|u\|)/s$ when $u \neq \theta$ [James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc., vol. 61, 1947, Ths. 4.1 and 5.1]. Let x be such that $ABx = x$. Suppose there exist elements $u \in \mathfrak{A}$ and $v \in \mathfrak{B}$ with $\|u-v\| < \|x-Bx\|$. Then for $0 \leqq s \leqq 1$,

$$\begin{aligned} \|x - Bx\| &\leqq \left\| (x - Bx) + \frac{s}{1+s}(Bx - v) \right\| \\ &= \left\| \frac{s}{1+s}(u - v) + \frac{1}{1+s}[(x - Bx) - su - x)] \right\| \\ &\leqq \frac{s}{1+s}\|u - v\| + \frac{1}{1+s}\|(x - Bx) - s(u - x)\|. \end{aligned}$$

Then $\|(x-Bx)-s(u-x)\| - \|x-Bx\| \geqq s(\|x-Bx\| - \|u-v\|)$ and

$$\lim_{\bar{s} \to 0-} \frac{\|(x - Bx) + \bar{s}(u - x)\| - \|x - Bx\|}{\bar{s}} \leqq \|u - v\| - \|x - Bx\| < 0.$$

But $\|(x-Bx)+s(u-x)\| \geqq \|x-Bx\|$, since $ABx = x$. Hence

$$\lim_{s \to 0+} \frac{\|(x - Bx) + s(u - x)\| - \|x - Bx\|}{s} \geqq 0.$$

Thus (iii) is not satisfied. Conversely, if some point of the unit sphere does not have a unique supporting hyperplane, then there is a two-dimensional subspace whose unit sphere has a point x with two supporting lines. Clearly $\mathfrak{A}$ and $\mathfrak{B}$ can be taken as intersecting supporting lines at x and at $-x$, respectively, so that $Bx = -x$ and $A(-x) = x$. But dist $(\mathfrak{A}, \mathfrak{B}) = 0$.

Also solved by C. C. Farrington and W. G. Strang, D. M. Friedlen, D. S. Greenstein, John Rainwater, Peter Treuenfels, and the proposers.

An Enumeration of the Rationals in (0, 1)

4851 [1959, 515]. *Proposed by Z. A. Melzak, McGill University, Montreal*

Let X be the set of all rational numbers of the open interval (0, 1). Prove that for some enumeration $\{x_n\}$ of X, the series $\sum_{n=1}^{\infty} \prod_{i=1}^{n} x_i$ diverges.

Solution by S. J. Einhorn, University of Pennsylvania. Let $\{b_n\}$ be the sequence $1/2, 2/3, 3/4, \cdots$, let $\{a_n\}$ be any enumeration of the rationals in $(0, 1)$ excluding those of the form $n/(n+1)$, and let $\sum c_n$ be an arbitrary divergent series of positive terms.

Then, because of the divergence of the harmonic series, an integer k_1 can be chosen large enough so that

$$d_1 = a_1 + a_1b_1 + a_1b_1b_2 + \cdots + a_1b_1b_2 \cdots b_{k_1} \geqq c_1.$$

k_2 can now be chosen large enough so that

$$d_2 = a_1b_1b_2 \cdots b_{k_1}(a_2 + a_2b_{k_1+1} + \cdots + a_2b_{k_1+1}b_{k_1+2} \cdots b_{k_2}) \geqq c_2.$$

Similarly let $k_3, k_4, \cdots$ be chosen.

Then the sequence $a_1, b_1, b_2, \cdots, b_{k_1}, a_2, b_{k_1+1}, \cdots, b_{k_2}, a_3, \cdots$ possesses the desired property since the series of the proposal contains the terms d_n and these already diverge by comparison with $\sum c_n$.

Also solved by N. J. Fine, Leopold Flatto, Fritz Herzog, S. S. Page, John Rainwater, P. L. Renz, J. T. Rosenbaum, R. T. Sandberg, Robert Spira, and the proposer.

Editorial Note. The proposer indicates a slightly more general result, similarly proved: If X is any countable set, dense in $(0, 1)$, and $\sum a_n$ is a divergent series of positive terms, then there is an enumeration $\{x_n\}$ of X, such that the series $S = \sum_{n=1}^{\infty} a_n \prod_{i=1}^{n} x_i$ diverges.

Renz relates that Professor Besicovitch gave one of his classes the similar problem: Let $\{r_i\}$ be an enumeration of the rationals in $(0, \infty)$. $\{r_i\}$ can be so chosen that $\sum_{i=1}^{\infty} \prod_{j=1}^{i} r_i$ either converges or diverges.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Elements of Calculus and Analytic Geometry. By George B. Thomas. Addison-Wesley, Reading, Mass., 1959. x+580 pp. $7.50.

This book is a revised and reorganized version of the author's successful *Calculus and Analytic Geometry* (see this MONTHLY, vol. 61, 1954, pp. 435–437) intended for a year course rather than a three-semester one. The main topics omitted are partial differentiation, multiple integration, solid geometry and vectors, differential equations, and infinite series. The same level of rigor has been kept and the understandability has, if anything, been increased.

ROY DUBISCH
Fresno State College

Elements of the Theory of Functions and Functional Analysis, Volume 1. *Metric and Normed Spaces*. By A. N. Kolmogorov and S. V. Fomin. Translated by L. F. Boron. Graylock Press, Rochester, N. Y. 1957, 129 pp., $3.95.

The book being reviewed has been derived from courses at the Moscow State University. The book is addressed to those with a background which may seem strange to us. For, while it begins with elementary concepts of sets and cardinal numbers, it supposes (for proofs left to the reader) that the reader knows the Weierstrass approximation theorem and the limitations of the Riemann integral. On the whole, the book is elegantly, concisely, and yet understandably written. The topics covered include, in addition to those mentioned, the standard basic material on metric spaces, contraction mappings and their application to existence theorems of differential and integral equations, and continuous curves in metric spaces. The reader is also introduced to normed linear spaces, Banach spaces, and their conjugate spaces, to linear operators and the spectrum of an operator, and to the Fredholm theorems using operator equations. There is an addendum on generalized functions.

The treatment throughout is mathematically satisfying. At the same time, its importance to applications is noted. A considerable amount of material has been compressed into a small book by the omission of topics such as Hilbert space, the L_p spaces, and others which require Lebesgue integration. A large number of examples are included in spite of these limitations. For the most part, the terminology and notation is consistent with general use in this country. One difference is that the definition of a compact set does not require that the set be closed. In a few places, for example, Section 18, Theorem 7, a few more words would greatly simplify the reading. The translator has done a good job. The "in other words" following the definition of continuity is the only place noticed where a little roughness resulted in an inaccuracy. The publisher has attended to the matters of format, printing, figures and binding in an excellent fashion.

James C. Bradford
Abilene Christian College and
Chance-Vought Aircraft, Inc.

Modern Geometry. By C. F. Adler. McGraw-Hill, New York, 1958. xiv+215 pp. $6.00.

This text participates in the recent trend to reduce Euclidean Geometry to selected topics (Ceva-Menelaus, Harmonic, Inversion) to make room for notions from projective and non-Euclidean geometries. The material included may serve to whet the appetite of the beginner, but the treatment is inadequate to give the student confidence in any of the new notions. There are a few exercises and topical bibliographies.

Arthur Bernhart
University of Oklahoma

Introductory College Mathematics. By Thomas L. Wade. Wiley, New York, 1959. xiv+319 pp. $5.50.

This text is designed for a three-semester-hour precalculus course for freshmen who have mastered high school algebra and trigonometry. Essentially, it is a brief course in plane analytic geometry and elementary differentiation, developed and presented in the terminology and notation of sets and relations. The book is well organized and well written, with rigor suitable to the level of instruction being maintained throughout.

Approximately the first third (Chs. 1, 2) is devoted to basic theory in deductive reasoning, sets, relations and real numbers; nearly half (Chs. 3–6) is concerned with basic analytic geometry through conic sections, and including polar coordinates; and the remainder (Chs. 7, 8) deals with functions and limits, differentiation and tangents. Inequalities are stressed throughout.

All chapters have a generous number of appropriate exercises. There is an exceptionally large number of well-drawn and clearly labeled figures, but there is no discernible difference between the "heavily drawn" and "lightly drawn" lines of Figures 145, 146.

H. S. KALTENBORN
Memphis State University

Fundamentals of Freshman Mathematics. By C. B. Allendoerfer and C. O. Oakley. McGraw-Hill, New York, 1959. 475 pp. $6.50.

This reviewer is inclined to say that this book is a new version of the authors' *Principles of Mathematics* written for a mathematically more conservative audience—teachers of potential scientists or engineers, and their students. Comparing it with *Principles* one might say it is three steps backward and two steps forward. On the whole the result is a book better suited to its intended audience. It retains the flavor of modern mathematics while ruthlessly suppressing proofs and omitting parts of *Principles*.

Many of the chapters from *Principles* are reprinted with the addition of routine drill problems (Ch. 2 on the number system, 10 on algebraic functions, 11 on exponents and logarithms, and 14 on analytic geometry). Others are rewritten versions of chapters in *Principles*, notably Chapter 1 which incorporates much of the material on logic and methods of proof, Chapter 6 on sets (Boolean Algebra and switching circuits are omitted), Chapter 8 on inequalities which was an appendix to the chapter on fields in the earlier version, and Chapter 9 on functions and relations. There are two chapters on trigonometry. Chapter 12 is new and is an impatient bow to old-fashioned trigonometry. For example, it includes a section *The Law of Tangents* which consists of one paragraph which does not even state the law of tangents. Chapter 13 is a cut-and-paste version of the trigonometry chapter in *Principles*.

Chapters 3, 4 and 5 are essentially review and drill on manipulative skills of "intermediate algebra" (such things as factoring, fractions, exponents and radicals).

Chapter 7 is a treatment of the application of matrices to the solution of simultaneous systems of linear equations.

Chapters 15 and 16 on "intuitive calculus" are also new and are especially well written for their intended audience. Chapter 17 on hyperbolic functions is regrettably more confusing than helpful. Many new ideas are mentioned casually with little real attempt at explanation; for instance, infinite series and complex exponents.

The spacing in Chapter 2 was apparently handled by editors who do not read copy. Bold faced headings such as **Closure Law of Addition** are separated from the paragraphs which explain them, and not separated from the following paragraph which is concerned with a different topic.

One disappointing lack of revision was the passage on the axiom of mathematical induction, page 32. The induction axiom is stated, and a rule of thumb is given which purports to test whether a given statement meets the conditions of the axiom, but which actually says nothing about the set of numbers mentioned in the axiom. This should have been rewritten.

JEAN M. CALLOWAY
Carleton College

A Modern Introduction to College Mathematics. By I. H. Rose, Wiley, New York, 1959, 530 pp. $6.50.

According to the author, this text book is intended for students who have had one and one half years of high school algebra and one year of high school geometry. Its intent seems to be to give the young student an understanding of the basic mathematical concepts and to acquire a "feel" for what mathematics is. The first part deals with the elements of set theory, natural numbers, real numbers and a little elementary algebra. The second part is concerned with analytic geometry and has a fairly extensive treatment of trigonometry. The third part has a very little of the elements of calculus, probability and statistics. Throughout the text there is a good deal of historical material either briefly mentioned or extensively treated. This book suffers—in common with many "modern" introductory texts—from attempts at complete rigor on the basis of fuzzy intuitive notions. Thus at one place the author relies on a figure and the properties of similar triangles which are defined twenty-four pages later, while at another he disparages the given proof because it relies on a figure and gives a "more rigorous" proof which also relies on a figure. Angle is defined in a rather bizarre fashion while area, which is used in the proof of the Pythagorean theorem, is not defined. Despite many such faults the book achieves its main purpose; the young student will not fail to see the power and the beauty of mathematics. The book is highly readable, the type is clear, the figures well drawn and the one or two misprints are very minor.

M. S. KNEBELMAN
Washington State University

Intermediate Algebra for the College Student. By J. Vincent Robison. Van Nostrand, Princeton, N. J., 1959. vii+286 pp. $4.25.

The author claims some 15 years' experience teaching intermediate algebra to college students, and has found ways of explaining points which should be satisfactory to most students. Proofs are given (or stated to be beyond the scope of the text), and ideas are discussed. However, in 43 places there are boxes containing "GUIDE TO . . . ," and giving step-wise procedures for accomplishing 43 different objectives. Thus, it would be possible to use the text as a cook book by ignoring the explanations and concentrating on the guides. It might be more difficult to persuade students to ignore the guides and concentrate on the explanations. On pages 48–69, 10 different types of word problems are discussed, one type for each set of exercises. The first set of word problems of various (mixed) types appears on pages 114–116. There is no review set of problems on factoring—each set of exercises is concerned with only one type of factoring. Topics beyond quadratic equations which are discussed include logarithms, variation and proportion, progressions and binomial theorem.

The typography is pleasing, and few misprints were noted. The book contains some statements to which a purist might object: (p. 6) The statements of the commutative and associative laws in words may create the wrong impression. (p. 99) "Fractions cannot be added unless they have the same denominator." (p. 129) "The left member [of $3x-4y=12$] states that $3x-4y=12$." This is a better-than-average text for teaching manipulation.

B. H. Arnold
Oregon State College

Einführung in Theorie und Andwerdung der Laplace-Transformation. By Gustav Doetsch. Birkhäuser Verlag, Basel, 1958. 301 pp. SFr. 39.40. (About $9.25).

This book, with German text, contains the majority of the topics to be found in the author's earlier book on the same topic. The style of the present work is considerably more direct, however, even though the motivations remain obscure. There has been a definite effort toward appealing to the interest of engineers and physicists, as well as mathematicians, and the book is designed as a text, rather than a reference. The exposition is clear and concise, and the development of the material follows in easy steps.

On the whole, one obtains the impression of a rigorous treatment of a classical tool applied to classical problems. The Laplace transform is presented as an operator and as a transformation. Among the topics considered are differential equations, both ordinary and partial, integral equations and difference equations, asymptotic developments, and some Tauberian theorems.

For use as a text book the only prerequisite would be a first course in complex variables, and the level would be senior year, or first year graduate.

Gordon Latta
Stanford University

Freshman Mathematics. By Paul K. Rees. Prentice-Hall, Englewood Cliffs, N. J., 1959. 285 pp. $5.25.

The preface states "This book is intended primarily for prospective elementary teachers and nonscience majors in arts college; however, it can be used by those students who need a refresher course in algebra or who must study algebra as a background for work in the mathematics of finance." To the reviewer this is a rather ambitious program.

The author has succeeded in producing a book for students with little or no background in high school mathematics. Ten of the seventeen chapters deal with junior high school mathematics or elementary high school algebra. The other seven chapters are on such topics as trigonometry, straight line, simple annuities, statistics, permutations and combinations. The reviewer believes that this book would be a suitable one for a refresher course. The few new topics might be sufficient to hold the interest of this type of student.

The reviewer *does not* agree that this is a book for elementary teachers and nonscience majors in arts college. These people should be taught more of the concepts, rather than a collection of techniques and manipulations.

JOHN C. KNIPP
University of Pittsburgh

Integral Equations. By I. G. Petrovskii. Graylock Press, Rochester, N. Y., 1957. 97 pp. $2.95, cloth, $1.95, paper.

As the size of this book readily indicates, Petrovskii's aims in his discussion of integral equations are of a more modest nature than those published by his Russian colleagues during the last decade or so. His primary purpose is to prove the existence of the solutions to linear integral equations of the second kind under sufficiently general conditions. No attempt has been made to consider the numerous applications to which this theory has been put; for this reason, this fine work may be less attractive to the physicist, engineer, and technician than to the mathematician. With the exception of one section of the appendix, no knowledge of Lebesque integration theory is required. The material is presented with great care and in accordance with present standards of mathematical rigor.

In Chapter 1 the author formulates the Fredholm theorems from known theorems of linear algebraic equations (by means of an analogy between linear integral and linear algebraic equations), and then proceeds to prove these theorems for certain classes of equations. Since the number of independent variables plays no essential role here, all proofs are carried out for an arbitrary number of independent variables. The procedure used by Petrovskii parallels the one originated by E. Schmidt. Firstly, the theorems are verified for the class of integral equations with the so-called degenerate kernels—the Pincherle-Goursat kernels—followed in order by an extension to continuous kernels of sufficiently small absolute bound, almost degenerate kernels, and to uniformly continuous kernels. The chapter concludes with a consideration of kernels of the form

$K(P, Q)/PQ^{\alpha}$ where points P and Q belong to a closed and bounded domain, K is a continuous function of (P, Q), PQ is the distance between P and Q and $\alpha < n$ (dimension of the domain). Chapter 2 consists of a three page discussion of the Volterra integral equations, while the last chapter is devoted to the Hilbert-Schmidt theory of integral equations with symmetric kernels. Here all functions are assumed to be real-valued and square integrable in every bounded region where they are defined.

The material is presented in a very readable and systematic manner, for which a good share of the credit should go to the translators Kamel and Komm. In the opinion of the reviewer, this book (actually a series of lectures) is not suitable as a text for normal course work; however, the graduate student in mathematics, who is primarily interested in the theory rather than application of integral equations, will find this an excellent presentation.

D. TRIFAN
University of Arizona

Calculus with Analytic Geometry. By Angus E. Taylor. Prentice-Hall, Englewood Cliffs, N. J., 1959. xv+762 pp. $8.50.

This is an excellent text for use in a first course in analytic geometry and the calculus. It is successful in presenting a union of the two subjects without slighting either.

A feature which will please those who believe that a course in analytic geometry and the calculus should be more than a study of techniques and type problems is the group of "Review Questions and Problems" which appears at the ends of every two or three chapters. These groups consist of thought-provoking questions concerning fundamental concepts and definitions, theory, and additional problems involving topics in the preceding chapters. These questions will keep the student (and the instructor) from forgetting that this is a course in mathematics.

Such a usually passed-over concept as the completeness of the real number system is treated by means of a sufficiently rigorous for its purpose but elementary discussion of Dedekind sections.

Physics and engineering departments will like the early introduction of the derivative (p. 36) and the antiderivative (p. 73). They also will like the emphasis on the applications of the calculus to theoretical mechanics as well as the use of vector ideas and notation, including an introduction to vector algebra and calculus.

The author has been successful in presenting a substantial course in the subject matter in a digestible but not predigested form.

LLOYD L. LOWENSTEIN
Arizona State University

An Introduction to the Theory of Groups. By P. S. Alexandroff. (Translated by Hazel Perfect and G. M. Petersen from the German edition, a translation from the original Russian book.) Hafner, New York, 1959. 112 pp. $3.25.

The author states in the foreword that "Every pupil in a senior class of a grammar school who enjoys doing mathematics is capable of grasping the idea of a group if he is interested and industrious. And so this book has been written in the first place for the mathematically inclined pupils in the senior classes in the grammar school, but also for those who teach mathematics to the senior or to the advanced level." This is a clear and well-motivated book on elementary group theory. The author takes great pains in defining each new concept and illustrating its meaning. Quite naturally, there are many interesting examples of groups of symmetries taken from geometry. An unusual feature of the book is that the group operation is designated by "+." Topics covered go through difference groups.

R. E. JOHNSON
Smith College

Three Pearls of Number Theory. By A. Ya. Khinchin. (Translated by F. Bagemihl, H. Komm, W. Seidel.) Graylock Press, Rochester, N. Y., 1952. 64 pp. $2.00.

This 64-page booklet was written in 1945 in the form of a letter to a wounded soldier recuperating in a Russian hospital. Of the recipient, not much is known except that he listened to Khinchin's lectures for one year before the war. The author was, of course, one of the most eminent Russian mathematicians. (He died in 1957.) His principal field of interest was mathematical probability, but he also made significant contributions in metrical number theory, among other subjects, and the present booklet attests to his interest in arithmetical questions.

The three pearls of the title are three theorems the author selected as especially suited for study by a partially trained young man, both because of the elementary nature of their proofs and because they were all first proved by young men after strenuous efforts on the part of more experienced mathematicians.

The first is van der Waerden's theorem on arithmetic progressions: given any two positive integers k and l, there exists an integer $n=n(k, l)$ such that, if an arbitrary segment of length n of the sequence of integers is divided in any manner into k classes, then an arithmetic progression of length l appears in at least one of the classes. The proof given here is due to M. A. Lukomskaya.

For an arbitrary increasing sequence A of positive integers $a_1, a_2, \cdots$, let $A(n)$ be the number of the a_i not exceeding n. The quantity g.l.b. $A(n)/n$ is called the Schnirelmann density of the sequence A, and is designated by $d(A)$. If A and B are two such sequences, the sum $A+B$ is the increasing sequence of all integers which can be written in one of the forms a_i, a_i+b_j, b_i. The second pearl is the theorem that in all cases, $d(A+B) \geqq \min(1, d(A)+d(B))$. This was

conjectured by L. G. Schnirelmann and E. Landau in 1931, and after many partial results on this so-called "$\alpha+\beta$ hypothesis" (including a proof in the special case $A=B$ by Khinchin himself) it was finally proved in 1942 by H. B. Mann. The proof exposed here is the simplification by E. Artin and P. Scherk.

The third pearl is the elementary proof discovered by E. Linnik in 1942 of Hilbert's theorem on Waring's problem: to every positive integer n there corresponds an integer k such that every natural number m can be represented as a sum of k nth powers of nonnegative integers, $m=x_1^n+\cdots+x_k^n$. Hilbert proved this theorem analytically in 1909, and estimates were obtained later for the size of k as a function of n, by analytic methods, but Linnik succeeded in constructing a completely elementary proof, based on an idea of Schnirelmann.

Almost no formal background is required for reading this book, and the author has taken great pains to make both the details and the over-all structure of the proofs transparent. On the other hand, there are intrinsic difficulties in the problems discussed, and the reader must add his own efforts to those of the author. The final paragraph of the book is worth quoting: "This proof [Linnik's], so exquisitely elementary, will undoubtedly seem very complicated to you. But it will take you only two or three weeks' work with pencil and paper to understand and digest it completely. It is by conquering difficulties of just this sort, that the mathematician grows and develops."

W. J. LeVeque
University of Michigan

Analytical Elements of Mechanics, Vol. I. By Thomas R. Kane. Academic Press, New York, 1959. xv+250 pp. $4.75.

The book is the first part of an introduction to classical mechanics, dealing only with the concepts of statics. An attempt is made to present the subject matters on a general basis and encourage the student to deduce the more special cases from this. Following a detailed exposition of elementary vector algebra, vector methods are used throughout the text. The theoretical aspects are more rigorously developed than usually found, including proofs and derivations in considerable detail. The emphasis then centers on transmitting a thorough understanding of the principles rather than proficiency in routine problem solving. Complemented by classroom communication, the book should prove a valuable help for the teaching of elementary mechanics.

H. W. Bergmann
University of Oklahoma

Engineering Statistics. By A. H. Bowker and G. J. Lieberman. Prentice-Hall, Englewood Cliffs, N. J. 1959. 585 pp. $8.25.

This book is a welcome addition to the material available for teaching statistics to physical scientists. It presents the topics in a manner which is more acceptable to statisticians than some other recent publications have done. The authors bring the statistical concepts forward rather than attempting to skirt

them, as so many books do that are aimed at "nonstatisticians." The book, however, is meant as a first course in statistics.

In certain places, the authors have presented the theory of a topic in a section immediately following the general discussion. This should allow more flexibility in the level of the class presentation.

Each chapter has ample problems. Most problems emphasize applications of theory to numerical examples, but a few theoretical problems are included.

The first four chapters cover the introductory concepts of random variables and distribution theory, including the normal distribution and its related distributions. The operating characteristic curve is introduced in chapter five. The next chapters deal with tests of hypotheses, estimation, fitting of straight lines and the analysis of variance. The chapter on analysis of variance includes discussion of fixed and random models for the one- and two-way classification. The *OC* curve and the computational procedures for these models are presented in detail. The book does not attempt to present additional material on experimental designs; latin squares, etc., do not appear, nor do the recent developments of response surface exploration.

Although this reviewer does not like the heuristic argument presented for the analysis of variance (it is easy to dislike such arguments) and feels that more should have been said about randomization (one could still get the impression that the validity of experimental results depend upon a model), I feel that this is a good book and well worth considering as a text for a course in engineering statistics.

W. H. WILLIAMS
McMaster University

Statistics: An Introduction. By D. A. S. Fraser. Wiley, New York, 1958. ix +398 pp. $6.75.

The preface states: "This book was prepared as a text for a two-semester course in mathematical statistics for those with the background of a full year of calculus." Such students could handle comfortably the first 35 pages which present an excellent introduction to probability. But the demands upon mathematical maturity become progressively more severe. On page 38 the reader is called upon to understand the distinction between a countable set of values and a continuum. On page 135 (a chapter on vectors and matrices comes later) during an incidental discussion of degrees of freedom one reads: "Geometrically, if we represent a sequence as a vector in N space, the degree of freedom is the dimension of the space the vector is free to move in." On page 365 we are told: "The theory of Galois fields can be used to construct orthogonal Latin squares whenever the number of levels is a prime or a power of a prime." The treatment of statistical inference, based largely on the work of R. A. Fisher, stresses analysis of variance, linear models, orthogonal transformations, pivotal solutions, and pivotal quantities. Each chapter ends with a bibliography and a wide assortment of problems many of which introduce essential background material before

posing a numerical problem. Problem 6, page 127, introduces coding; problem 22, page 130, contains the only discussion of quality control in the book. By sprinkling bits of theory and partially defined terms among the problems the author has included a surprising amount of material in a book of moderate size.

Mathematics majors should find the book stimulating and rewarding but in the opinion of the reviewer most engineering students would find it frustrating.

V. V. LATSHAW
Lehigh University

The Growth of Mathematical Ideas, Grades K–12. (Twenty-fourth Yearbook, The National Council of Teachers of Mathematics) Phillip Jones, Editor. Washington, D. C., 1959. x+507 pp. $5.00.

This yearbook has three basic aims: to highlight "the essential elements of those basic mathematical understandings which should be continually developed and extended throughout the entire mathematics curriculum, grades K–12 and beyond," "to define and illustrate some classroom procedures and their psychological bases," and "to assist teachers and supervisors to extend and apply the ideas of the book to their own situations."

The "basic mathematical understandings" are included in Chapters 2–8 inclusive. The chapter titles are in order: Number and Operation, Relations and Functions, Proof, Measurement and Approximation, Probability, Statistics, and Language and Symbolism in Mathematics. These topics were selected by the Yearbook Committee as "the most basic mathematical themes which should be central to the entirety of a modern mathematics curriculum." The Committee recognized that there are other ideas and concepts which belong in the mathematics programs of the schools, but lack of space did not permit including them.

The topics and concepts are discussed as continuing themes and hence each chapter is designed so that teachers from the elementary school through the high school will find useful material for their particular level of teaching. It seems to this reviewer that the authors have accomplished this purpose well and that this feature of the yearbook is probably its more significant contribution.

Only three chapters are devoted to the second and third aims of the yearbook. Their titles are: Mathematical Modes of Thought, Implications of the Psychology of Learning, and Promoting Continuous Growth of Mathematical Concepts. Since these chapters deal with procedures as contrasted to concepts and ideas, it seems probable that this part of the book will be less useful to teachers than the first part. This point of view is based on the assumption that the most urgent need of present day teachers of mathematics at all levels of the schools is more knowledge of mathematics, especially of the more modern concepts.

As stated in the preface it is inevitable that this yearbook "should be compared with and related to the twenty-third yearbook, *Insights Into Modern*

Mathematics." The yearbooks did not have the same purpose, but they should supplement each other very well. The twenty-third yearbook stressed theory and modern concepts, while this yearbook stresses continuing themes of basic mathematical ideas so that these may be recognized by pupils and teachers as a part of growth and development of mathematical understandings.

JAMES H. ZANT
Oklahoma State University

Japanese Chess, The Game of Shogi. By E. Ohara. Bridgeway (Tuttle) Press, 1958. 182 pp. $2.75.

Many western mathematicians have been fascinated by the game of Go but Japan's other great strategic game, Shogi, seems almost unknown here. This little manual should help remedy this imbalance. Shogi, unlike Go, is clearly a form of Chess and similarities to western chess as well as basic differences are clearly pointed out. Chess players particularly should find Shogi, as here presented, both easy to learn and interesting to play.

R. C. SIMPSON
St. Lawrence University

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

New Mexico State University: Dean Earl Walden, Head of the Mathematics Department and Dean of the Graduate School, will become full-time Dean of the Graduate School on July 1. Professor R. B. Crouch will become Head of the Mathematics Department.

Oregon State College: Associate Professors H. E. Goheen and A. R. Poole have been promoted to Professors; Mr. Helmut Groemer has been promoted to Assistant Professor; Associate Professor W. M. Stone is on leave at the Boeing Airplane Company, Seattle, Washington.

Dr. E. B. Allen, Head of the Department of Mathematics at Rensselaer Polytechnic Institute, has been appointed Dean of the Graduate School, to succeed Dr. C. O. Dohrenwend.

Professor Cristobal de Losada y Puga, Catholic University of Peru, has been incorporated as "miembro de numero" in the Academia Peruana correspondent of the Real Academia Espanola. He has also been elected Director of Publications in the Academia Nacional de Ciencias Exactas, Fisicas y Naturales de Lima.

Miss Leah Fine, Lockheed Aircraft Corporation, Marietta, Georgia, has accepted a position as Programmer-Analyst with the System Development Corporation, Paramus, New Jersey.

Professor R. M. Foster, Polytechnic Institute of Brooklyn, was awarded the honorary degree of Doctor of Science by Fairleigh Dickinson University on January 15, 1960.

Mr. J. W. Fothergill, Jr., Aerojet General Corporation, Nimbus, California, has accepted a position as Senior Programmer with the Thiokol Chemical Corporation, Brigham City, Utah.

Associate Professor Herta T. Freitag, Hollins College, has been promoted to Professor.

Dr. H. D. Friedman, General Electric Company, DeWitt, New York, has accepted a position as Mathematician with Technical Operations, Inc., Burlington, Massachusetts.

Mr. W. G. Hazlett, System Development Corporation, Santa Monica, California, has accepted a position as Scientific Computer Analyst with Temco Aircraft, Garland, Texas.

Associate Professor I. N. Herstein, Cornell University, has been promoted to Professor.

Professor M. W. Keller, Purdue University, has been promoted to Assistant Head of the Department of Mathematics and Statistics in charge of Extension and graduate teacher programs.

Associate Professor R. E. Lee, The University of Missouri School of Mines and Metallurgy, has been promoted to Professor.

Mr. Parviz Morewedge, Douglas Aircraft, Santa Monica, California, has accepted a position as Logical Designer with Litton Industries, Beverly Hills, California.

Mr. M. J. Schrader, Army, Redstone Arsenal, Alabama, has accepted a position as Mathematician with McDonnell Aircraft Corporation, St. Louis, Missouri.

Mr. D. B. Singer, Armour Research Foundation, Illinois, has accepted a position as Research Specialist with the Missiles and Space Division of Lockheed Aircraft Corporation, Van Nuys, California.

Mrs. Dorothy P. Smith, Kansas State College, has been appointed Instructor at New Mexico Highlands University.

Mr. Arthur Sullivan, Army Research and Development Laboratory, Fort Monmouth, New Jersey, has accepted a position as Senior Microwave Engineer with the Canoga Division of the Underwood Corporation, Van Nuys, California.

Mr. D. H. Taranto, International Business Machines Corporation, Flushing, New York, has been appointed Instructor at Carleton College, Northfield, Minnesota.

Mr. Neal Bradley, Georgia Institute of Technology, died December 20, 1959. He was a member of the Association for three years.

Mr. G. B. Glover, Rutgers College of South Jersey, died December 3, 1959. He was a member of the Association for five years.

Mr. C. E. Harrington, W. H. Smith, Inc., Springville, New York, died January 7, 1960. He was a member of the Association for thirty-eight years.

Professor T. W. Moore, U. S. Naval Academy, died December 17, 1959. He was a member of the Association for thirty-five years.

Mr. V. E. Roberts, University of Alaska, died January 2, 1960. He was a member of the Association for seven years.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1960:

University of Buffalo, June 27 to August 6: Professor Schneckenburger, higher algebra, functions of a complex variable. NSF Institute: Professor Montague, introduction to modern algebra; Professor Olson, probability and statistics, fundamental concepts of calculus; Mr. Montzingo, demonstration class and workshop.

University of California, Berkeley, June 20 to July 29: Professor Namioka, foundations of analysis; Professor Dubisch, foundations of mathematics. August 1 to September 9: Professor Godemont, introduction to the theory of functions of a complex variable; Professor Freudenthal, Lie groups.

University of Colorado, June 20 to August 26: Dean Crawford (Mount Allison University) and Professor Zirakzadeh, higher algebra and analysis for teachers; Professor Halfar (University of Nebraska), foundations of analysis and modern geometry; Professor Nering (University of Arizona) theory of numbers and probability theory. June 20 to July 22: Professor Briggs, infinite processes. July 25 to August 26: Professor Emeritus Kempner, history of mathematics.

Columbia University, Teachers College, July 5 to August 12: Dr. Allegri, teaching of arithmetic, history of mathematics; Professor Fehr, supervision and problems in teaching secondary school mathematics; Professor Felix and Dr. Steiner, elements of modern mathematics; Professor Rosskopf, logic in elementary mathematics; Mr. Rourke, introduction to analysis, professionalized subject matter for senior high school mathematics; Dr. Zlot, teaching of algebra, elements of algebra.

Cornell University, June 27 to August 6: Professor Robison, determinants and matrices; Mrs. Hertzig, advanced calculus.

De Paul University, (Day): Professor DeCicco, theory of numbers, solid analytic geometry; Dr. O'Neil, Taylor series. (Evening): Dr. Yao, theory of rings.

University of Florida, June 21 to August 13: Professor Blake, introduction to mathematical thought; Professor South, mathematical statistics; Professor Lang and Professor Patterson, advanced mathematics for engineers and physicists; Professor Hadlock, advanced topics in calculus; Professor Moore, introduction to topology; Professor Pirenian, vector analysis; Professor Hutcherson, synthetic projective geometry; Professor Phipps, foundation of geometry, history of elementary mathematics; Professor Cowan, partial differential equations.

Indiana University, June 17 to August 12: Professor Wolfe, elementary statistics, projective geometry; Professor MacKenzie, introduction to modern algebra, topics for mathematics teachers; Staff, history of mathematics, introduction to probability theory, mathematical reading and research.

University of Kansas, June 13 to August 6: Professor Himmelberg, elementary number theory, topological groups; Professor Kruse, applied math I, applied math II; Mr. Pulley, programming for a digital computer.

Kent State University, June 20 to July 23: Selected topics for classroom teachers; solid analytic geometry; introduction to modern algebra; history of mathematics. July 25 to August 27: Vector analysis; mathematical statistics I; the teaching of junior high school mathematics.

Michigan State University, June 21 to September 2: Professor Herzog, sets and abstract spaces, boundary behavior problems in complex variables; Professors Larcher and Stewart, matrices and groups; Professor J. H. Powell, theory of polynomials; Professor Reid, boundary value problems; Professors Reid and Sheedy, differential equations; Professor Stelson, theory of equations. June 21 to July 28: Professor Campbell, theory of numbers; Professor Grove, projective geometry; Professor Hall, advanced calculus I and II; Professor Hill, concepts in algebra; Professor Larcher, complex variables I; Professor Norman, elements of complex variables, vector analysis; Professor Oehmke, group theory; Professor J. E. Powell, concepts in calculus; Professor J. H. Powell, differential equations. July 29 to September 2: Professor Hocking, advanced calculus II, topology III; Professor Kelly, set theory and axiomatic systems, college geometry; Professor Nordhaus, concepts in geometry, advanced calculus III; Professor J. H. Powell, introduction to matrices; Professor Stewart, rings and algebras; Professor Wells, partial differential equations, complex variables II.

University of Minnesota, College of Science, Literature, and the Arts, June 13 to July 16: Professor Engeler, elementary set theory; Professor Gil de Lamadrid, differential equations, topics in topology; Dr. Joichi, advanced algebraic theory I, solid analytic geometry. July 18 to August 20: Professor Harper, advanced algebraic theory II; Professor Hatfield, probability; Dr. Miracle, intermediate differential equations, theory of numbers. An institute for high school teachers of mathematics will be conducted from June 13 to August 6.

University of Nebraska, July 8 to July 31: Professor Abel, topics in geometry; Professor Basoco, Laplace transforms; Professor Harris, topics in probability and statistics; Professor Miller, topics in algebra.

University of North Carolina, June 9 to July 19: Professor Cameron, fundamental concepts of mathematics with special reference to algebra; Professor Garner, history of mathematics; Professor Hill, elementary mathematical statistics; Professor Jones, advanced calculus I; Professor Linker, differential equations. Professor Mac Nerney, seminar in analysis. July 20 to August 27; Professor Carr, introduction to modern algebra I; Professor Hoyle, advanced calculus II; Professor Lasley, analytic geometry from a higher standpoint; Professor Mackie, theory of equations; Professor Pettis, topics in analysis; Professor Wells, seminar in analysis.

Northwestern University, June 21 to August 13 and June 21 to July 30: Engineering mathematics 1-T; numerical methods; digital computer programming 1 and 2; statistics for teachers; probability; algebra for teachers; geometry for teachers; the history of mathematics I; foundations of calculus for teachers; complex variables for applications; topics in modern mathematics for teachers; modern algebra; introduction to topology.

University of Oklahoma, June 6 to August 6: Professor Bernhart, college geometry; Professor Brixey, principles of mathematical statistics; Professor LaFon, vector analysis; Professor Levy, foundations of analysis; Professor Nicol, theory of numbers; Professor Pan, elementary differential equations; Professor Springer, analytic projective geometry, ordinary and partial differential equations.

University of Pennsylvania, June 27 to August 5: Professor Schub, introduction to real analysis, matrix theory and numerical calculations; Professor Abian, numerical methods and differential equations; Professor Gottschalk, theory of sets.

Syracuse University, July 5 to August 12: Professor Gould, higher mathematics for engineers; Professor Cargo, foundations of geometry; Professor Hemmingsen, topology; Professor Exner, logic for secondary teachers; Professor Pollak, analysis and application I (differential equations).

University of Washington, June 20 to July 20, July 21 to August 19: Differential equations; advanced calculus, linear algebra; introduction to modern algebra; fundamental concepts of analysis; topics in applied analysis; foundations of geometry; projective geometry; complex variable.

West Virginia University, June 6 to July 15: Professor Vest, differential equations, theory of probability; Professor Bragg, topology, history of mathematics; Professor Cochran, advanced calculus. July 18 to August 26: Professor Cunningham, foundations of algebra and analysis, topology; Professor Peters, advanced calculus, algebraic surfaces; Mr. Lowenberg, modern geometry for teachers.

University of Wisconsin, June 20 to August 13: Professor Kleene, modern algebra, differential geometry; Professor Bicknell, introduction to mathematical statistics; Professor Van Engen, survey in the foundations of arithmetic; Professor Spragens (University of Louisville), survey of the foundations of algebra; Professor Fox (Princeton University), theory of numbers, advanced topics in algebraic topology; Mr. Howard, applied differential equations; Staff, topics in geometry, applied mathematical analysis, introduction to the theory of probability, convex figures and inequalities, theory and operation of computing machines, modern views of mathematics, topics in the history of mathematics, foundations of mathematics.

University of Wyoming, June 13 to July 15: Professor Walsh, introduction to higher algebra, advanced calculus; Professor W. N. Smith, ordinary differential equations; Miss Guerin, college geometry; Professor Varineau, fundamental concepts of mathematics, numerical analysis. July 18 to August 19: Professor DeZur, theory of matrices and determinants; Professor S. R. Smith, vector analysis; Professor Schwid, partial differential equations, Fourier series and boundary value problems; Professor Barr, seminar in geometry.

INTERNATIONAL CONFERENCE

The Mathematics Research Center, United States Army, will conduct at the University of Wisconsin, June 7–15, 1960, an INTERNATIONAL CONFERENCE ON

PARTIAL DIFFERENTIAL EQUATIONS AND CONTINUUM MECHANICS. A considerable number of invited addresses will be given and there will be sessions for contributed papers. The invited addresses and abstracts of the contributed papers will be published as the proceedings of the Conference. Further information may be obtained from Professor R. E. Langer, Mathematics Research Center, U. S. Army, University of Wisconsin, Madison 6, Wisconsin.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1959

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1959. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the Buffalo office of the Association.

There was a surplus of $16,499 in the Current Fund of the Association for 1959, of which $15,000 has been transferred to the General Fund. The Treasurer is pleased to report gifts from two members of the Association amounting to $210, which have also been added to the General Fund, which serves as an endowment fund for the Association.

Assets of the Association	January 1, 1959	December 31, 1959
M & T Trust Co., Buffalo	$ 19,687.37	$ 25,081.05
Savings Accounts	108,363.39	111,699.22
Securities	169,620.23	275,542.00
	$297,670.99	$412,322.27
Funds of the Association		
Current Fund	$ 1,027.14	$ 2,526.07
Carus Fund	26,706.76	28,798.31
Chace Fund	8,760.58	9,545.72
Houck Fund	13,832.45	12,028.00
Chauvenet Fund	1,620.95	1,642.31
Dunkel Fund	19,451.65	19,744.78
General Fund	51,428.73	64,265.57
	$122,918.26	$138,550.76
Visiting Lecturers Fund	38,285.38	78,423.03
Fund for Committee on Undergraduate Program	88,691.79	54,923.28
Fund for Committee on High School Contests	427.34	Deficit 77.33
Fund for Survey of Non-Teaching Mathematical Employment	1,218.43	10,383.55
Secondary School Lecturers Fund	16,054.92	73,883.15
Fund for Committee on Production of Films	27,827.45	15,437.08
Fund for Washington Office	—	34,154.39
Fund for Survey of European Mathematical Education	—	6,644.36
	$297,670.99	$412,322.27

THE FORTY-THIRD ANNUAL MEETING OF THE ASSOCIATION

The forty-third annual meeting of the Mathematical Association of America was held at the Conrad Hilton Hotel, Chicago, Illinois, from Thursday to Saturday, January 28 to 30, 1960, in conjunction with meetings of the American Mathematical Society, the National Council of Teachers of Mathematics, and the Society for Industrial and Applied Mathematics. The Saturday sessions of the Association were joint sessions with the National Council of Teachers of Mathematics. There were registered 1322 persons, including 816 members of the Association.

Sessions of the Association were held on Thursday morning and on Friday morning in the Boulevard Room of the Conrad Hilton Hotel and on Saturday morning and afternoon in the Eighth Street Theater. Professor P. T. Bateman presided at the lectures on Friday morning. President C. B. Allendoerfer presided at the Annual Business Meeting and the general session on Saturday morning. The Program Committee for the meeting consisted of P. T. Bateman, Chairman; H. Flanders, P. S. Jones, and M. Kline.

FIRST SESSION OF THE ASSOCIATION

"Exterior Differential Forms in the Undergraduate Program." Moderator: Harley Flanders, University of California, Berkeley. "Introductory Remarks," by Carl B. Allendoerfer, University of Washington; "Exterior Differential Calculus," by Lawrence Markus, University of Minnesota; "Applications to Geometry," by Albert Nijenhuis, University of Washington; "Concluding Remarks," by H. Flanders, University of California, Berkeley.

SECOND SESSION OF THE ASSOCIATION

"The Four and Eight Square Problem and Division Algebras," by Charles W. Curtis, University of Wisconsin.

Annual Business Meeting and Award of the Chauvenet Prize.

"Entropy in Ergodic Theory," by Paul R. Halmos, University of Chicago.

JOINT MEETING OF MAA AND NCTM

General Session

"Short and Long Range Improvement of the School Mathematics Curriculum," by Paul C. Rosenbloom, University of Minnesota; "Better Curriculum or Better Pedagogy?" by Morris Kline, New York University.

Session on Advanced Placement

Moderator: Harold P. Fawcett, Ohio State University. "Collegiate Views and Experiences," by E. P. Vance, Oberlin College and W. T. Scott, Northwestern University; "High School Views and Experiences," by Edwin C. Douglas, The Taft School, Watertown, Connecticut, and Henry Swain, New Trier Township High School, Winnetka, Illinois.

Session on the Nature and Role of Geometry in High School

Moderator: Max Beberman, University of Illinois. "Collegiate Views," by Kenneth B. Leisenring, University of Michigan, and Robert R. Christian, University of British Columbia and University of Illinois; "High School Views," Henry W. Syer, Kent School, Kent, Connecticut, and Roderick McLennon, Arlington Heights High School, Arlington Heights, Illinois.

Session on Calculus in High School

Moderator: Phillip S. Jones, University of Michigan. "Collegiate Views," by Albert A. Blank, New York University, and J. H. Neelley, Carnegie Institute of Technology;

"High School Views," by W. Eugene Ferguson, Newton High School, Newtonville, Massachusetts, and Hubert Davis, Cranbrook School, Bloomfield Hills, Michigan.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Thursday evening in Private Dining Room 18 of the Conrad Hilton Hotel with twenty-six members present. Among the more important items of business transacted were the following:

The Board approved the appointment by President Allendoerfer of the following Nominating Committee for 1960: F. A. Ficken, Chairman; E. P. Vance, and Paul B. Johnson.

The Board approved the following schedule of future meetings: Michigan State University, August 29—September 1, 1960; Willard Hotel, Washington, D. C., January 25–27, 1961; Oklahoma State University, August 28–31, 1961; Kansas City, Missouri, January 1962; University of Colorado, August 26–29, 1963; University of Michigan, August, 1964.

The Board voted that the list price of all Carus Monographs be set at $2 per copy for members and $4 per copy for nonmembers for all orders received after April 1, 1960.

The Board also gave its approval to the recommendation of the Committee on Sections whereby each section will receive $60 per year plus an additional $10 per year for each 100 members or fraction thereof in excess of 200.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday, January 29, 1960 in the Boulevard Room of the Conrad Hilton Hotel, Chicago, Illinois. President C. B. Allendoerfer presided.

The Secretary announced the results of the balloting for officers in which 1886 votes were cast: Dr. A. S. Householder, Oak Ridge National Laboratory, was elected First Vice-President for the two-year term 1960–1961, and Professors R. C. Buck, University of Wisconsin, and J. G. Kemeny, Dartmouth College were elected Governors for the three-year term 1960–1962.

The membership of the Association was 9113 as of January 26, 1960, a net gain of 480 since August 17, 1959.

Professor G. B. Price reported for the Committee on the Undergraduate Program in Mathematics.

AWARD OF THE CHAUVENET PRIZE

The 1960 Chauvenet Prize was awarded to Professor Cornelius Lanczos of the Dublin Institute for Advanced Studies for his paper "Linear Systems in Self-Adjoint Form," published in this MONTHLY, vol. 65 (1958), pp. 665–679.

The 1960 Chauvenet Prize is awarded for a noteworthy expository paper published in English during the three-year period 1956–1958 by a member of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. This is the twelfth award of the Chauvenet Prize since its institution by the Association in 1925.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Wednesday, January 27, through Friday, January 29. The thirty-third Josiah Willard Gibbs Lecture, entitled "Quantum Field Theory," was given by Professor Julian Schwinger of Harvard University. Invited addresses were delivered by Professor A. S. Besicovitch of Cambridge University and the University of Pennsylvania, and Professor Paul Rosenbloom of the University of Minnesota.

The Society for Industrial and Applied Mathematics met on Friday, January 29, at which time Dr. David Slepian and Dr. H. O. Pollak of the Bell Telephone Laboratories, Inc., gave invited addresses.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: L. R. Wilcox, Chairman; H. L. Alder, H. J. Curtis, E. H. C. Hildebrandt, G. L. Walker, J. W. T. Youngs.

Registration headquarters was located in the third floor corridor of the Conrad Hilton Hotel. On Saturday only, registration took place in the lobby of the Eighth Street Theater. Sleeping accommodations were available in the Conrad Hilton Hotel and in other hotels. Meals were available in the Hotel and at numerous nearby restaurants and cafeterias. The Mathematical Sciences Employment Register was located in Parlor Dining Room 2 and textbook exhibits were located on the fourth floor of the Conrad Hilton Hotel.

Since the opportunities for entertainment and recreation throughout Chicago in general, and the Conrad Hilton Hotel in particular, are so extensive, it was decided not to hold an annual banquet for this occasion.

HENRY L. ALDER, *Secretary*

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The fourth annual meeting of the New Jersey Section of the Mathematical Association of America was held at Princeton University, Princeton, New Jersey, on November 7, 1959. Dr. H. O. Pollak, Chairman of the Program Committee, presided at the morning session and the business meeting. Professor A. W. Tucker of Princeton University presided at the afternoon session. About 115 persons were present, including 81 members of the Association.

The following officers were elected: Chairman, Dr. H. O. Pollak, Bell Laboratories, Murray Hill, New Jersey; Secretary-Treasurer, Professor I. L. Battin, Trenton State College; Member of the Executive Committee, Dr. G. Y. Cherlin, Mutual Benefit Life Insurance Company, Newark, New Jersey. Continuing members of the Executive Committee are Dr. J. D. Daugherty and Professor E. P. Starke.

Dr. Cherlin reported for the Contest Committee that 4,462 students from 133 schools had participated in the 1959 contest.

The following papers were presented, all by invitation:

1. *Atoms, trees, and algebraic numbers*, by Professor R. C. Buck, University of Wisconsin.

An observer stands at the origin in a forest of trees located at the lattice points (k, m). The radius of each tree is $F(r)$ where $r=(k^2+m^2)^{1/2}$. In this context, many of the classic problems in Diophantine approximation are embraced by the question: How far can one expect to see into the forest? Various solutions to this, ranging from Khintchine's theorem to Roth's theorem, were discussed, and results extended to the three-dimensional case.

2. *The transportation problem*, by Professor Murray Gerstenhaber, University of Pennsylvania.

In the transportation problem, a homogeneous commodity is available at m "producers" in amounts $a_1, \cdots, a_m$ and required at n "consumers" in amounts $b_1, \cdots, b_n$ where $\sum a_i = \sum b_j$. The cost of transporting a unit quantity from the ith producer to the jth consumer is given, and the problem is to determine the amount that should be delivered from the ith producer to the jth consumer to minimize the total cost of transportation. By transforming the problem into a model of a competitive economic situation, it has been possible to program UNIVAC I to solve problems with $m \leqq 180$, $n \leqq 2100$. With more recent machines, values of m and n up to 40,000 are both possible and feasible.

3. *The work of the School Mathematics Study Group*, by Professor E. G. Begle, Yale University.

The objectives and accomplishments of SMSG were reviewed. Teaching materials which the Group has developed for each of grades seven through twelve were described in detail. Attention was called to the work done in preparing reference materials for teachers, and special materials for gifted students. The combination of university faculty members, secondary school teachers, and mathematicians from business and industry in the writing groups has proven to be very productive.

I. L. BATTIN, *Secretary*

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The fifth annual meeting of the Northeastern Section of the Mathematical Association of America was held at Boston College, Chestnut Hill, Massachusetts, on November 28, 1959. Professor N. H. McCoy, Chairman of the Section, presided at the morning and afternoon sessions. There were 115 persons registered for the meeting, including 95 members of the Association.

Officers chosen for 1959–60 were Professor J. G. Kemeny, Dartmouth College, Chairman; Professor D. E. Christie, Bowdoin College, Vice-Chairman; Mr. R. S. Pieters, Phillips Academy, Secretary-Treasurer. Professor R. A. Rosenbaum of Wesleyan University was appointed liaison officer to deal with mathematical matters within the state of Connecticut. W. E. Ferguson of Newton (Mass.) High School gave a report of the secondary school visiting lectureship program.

The executive committee of the Section proposed a motion that a one-dollar registration fee be charged to all members attending the annual meeting of the section, for the year 1960. The motion was carried unanimously.

By invitation of the Executive Committee of the Section, the following hour talks were presented:

1. *On some problems of group theory*, by Professor Richard Brauer, Harvard University.

Some unsolved problems in the theory of groups of finite order are discussed. These deal with the finite collineation groups of a given degree, relations between the representations of the group by linear transformations and abstract properties of the group, the structure of group algebras, certain types of generalizations of the group concept, the simple finite groups, and related topics.

2. *The mean value property of harmonic functions*, by Professor Hans Rademacher, University of Pennsylvania. (By invitation).

3. *Some aspects of the theory of knots*, by Professor R. H. Crowell, Dartmouth College, introduced by the Secretary.

For any tame knot K in the 3-sphere S^3, there exists a tame orientable surface in S^3 whose boundary is K. The *genus* of K is the minimum of the genera of such surfaces. The following is due to H. Schubert (S.-B. Heidelberger Akad. Wiss. Math., Nat. Kl., 1949, pp. 57–104). For any two oriented tame knot types, an associative commutative product is defined, and the trivial type is an identity. The genus of a product is the sum of the genera of the factors. The existence of prime factorization follows. Prime factorization is unique and, since the identity is the only unit, the cancellation law holds.

4. *Decision making*, by Professor Howard Raiffa, Harvard University.

R. S. PIETERS, *Secretary*

CARUS MONOGRAPHS

In April, Carus Monograph No. 13 will be published. It is entitled "A Primer of Real Functions." The author is Professor Ralph P. Boas, Jr., of Northwestern University. A brief description of the Monograph is as follows:

This Monograph develops parts of the theory of sets and real functions for the sake of their intrinsic interest. The subject, which originally grew out of detailed examinations

of the concepts of continuity, differentiability, and the like, startled the mathematicians of the nineteenth century with nowhere-differentiable continuous functions and space-filling curves. Although it is now more usually studied as background material for complex analysis or for the theory of integration, it continues to produce surprises even today. The first part of the Monograph presents selected topics from the theory of sets in metric spaces and illustrates the power of the theory in applications to analysis. The second part applies the tools developed in the first part to a more systematic study of increasingly specialized classes of functions: continuous, differentiable, monotonic, convex and infinitely differentiable. Some of the results are old and familiar, at least by reputation; some are old and undeservedly forgotten; and some are quite recent.

Because of increases in the cost of printing and distribution, the Board of Governors has voted that the list price of all Carus Monographs will be set at $2 per copy for members and $4 per copy for non-members, for all orders received for Carus Monographs 1–4, 6–12, after April 1, 1960, and for all orders for Carus Monograph No. 13. Members should send orders to the Buffalo office of the Association.

CALENDAR OF FUTURE MEETINGS

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA, Earlham College, Richmond, May 7, 1960.

IOWA

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, May 7, 1960.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, St. Olaf College, Northfield, May 7, 1960.

MISSOURI

NEBRASKA

NEW JERSEY

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, University of Oklahoma, Spring, 1960.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6-7, 1960.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

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OCEANOGRAPHER
Application of Digital Computer techniques to Oceanographic Data Sensing and Processing.

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ON NUMERICAL APPROXIMATION is comprised of the twenty-one papers which were delivered at The Symposium on Numerical Approximation conducted by the Mathematics Research Center, United States Army, at the University of Wisconsin in April, 1958. The objective of this symposium was the presentation and discussion of recent developments in the field of numerical approximation. The papers are centered around three general themes: *Linear Approximation, Extremal Approximation,* and *Algorithms.*

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